Stochastic inflation and primordial black holes

Valencia, 14 May 2021

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Based on 2012.06551, in collaboration with D. Figueroa, S. Raatikainen, S. Räsänen

Concepts

Cosmic inflation

- Accelerating expansion of space in the early universe

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Cosmological perturbations

- Cosmic microwave background, ...

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Primordial black holes

- Dark matter candidate

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- Includes non-linear effects

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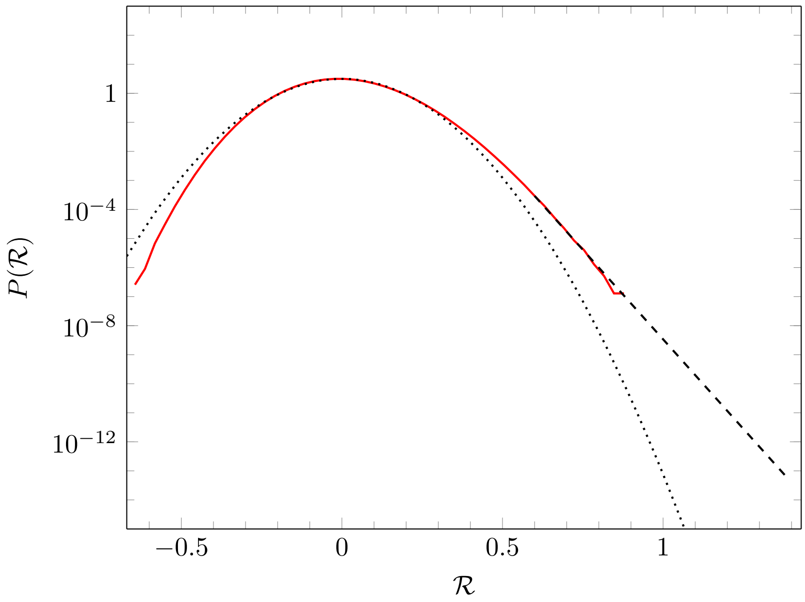
Stochastic inflation

- Includes non-linear effects
- Numerical method: even more non-linearities



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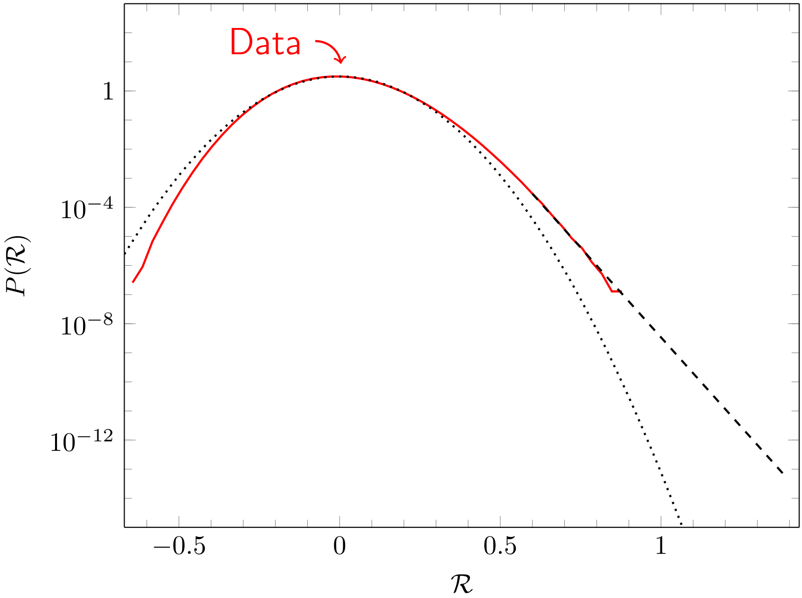
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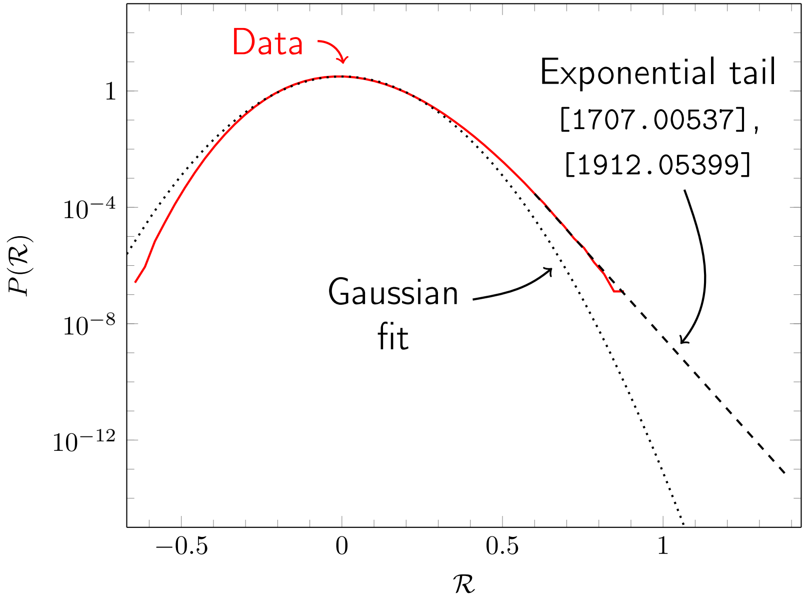
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[1912.05399]
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Explains origin of cosmological perturbations

Cosmic inflation with a scalar field

$\ddot{a}(t) > 0$ accomplished by scalar field matter

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$$3H^2M_P^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \quad H \equiv \frac{\dot{a}}{a}$$

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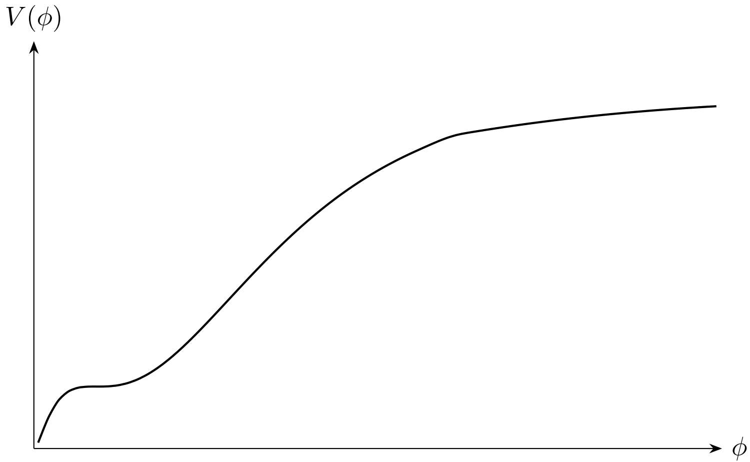
Inflation happens when $V(\phi)$ dominates over $\dot{\phi}^2$

Our example model is typical



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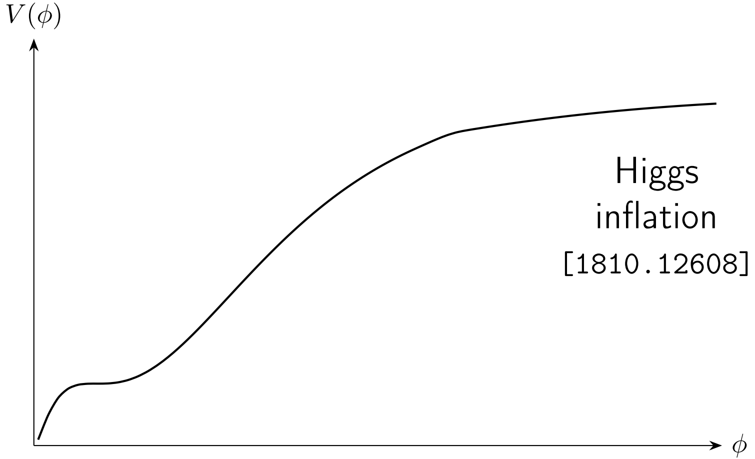

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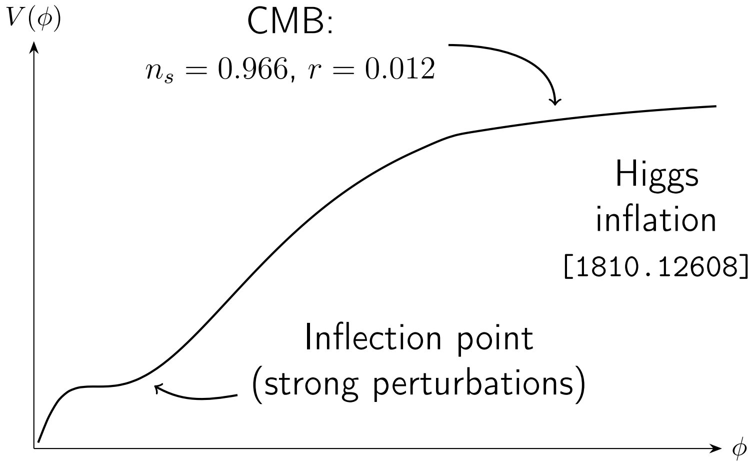

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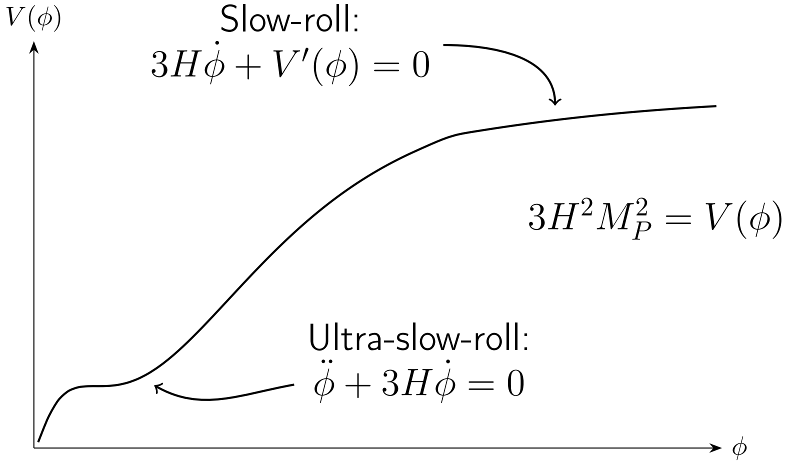

Perturbations depend on scale

Origin of perturbations: fluctuations of quantum vacuum

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Space expands and perturbations get stretched

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Strong perturbations from ultra-slow-roll inflation

Linear perturbation theory

Expand to linear order:

$$\delta\ddot{\phi}_{\vec{k}} + 3H\delta\dot{\phi}_{\vec{k}} + V''(\phi)\delta\phi_{\vec{k}} = 0$$

Comoving curvature perturbation and its power spectrum:

$$\mathcal{R}_{\vec{k}} = \frac{\delta\phi_{\vec{k}} H}{\dot{\phi}}, \qquad \mathcal{P}_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2}|\mathcal{R}_{\vec{k}}|^2$$

CMB observables in slow-roll

$$\mathcal{P}_{\mathcal{R}}(k) = A_s \left(\frac{k}{k_*}\right)^{n_s - 1}, \qquad A_s = \frac{V}{24\pi^2 \epsilon_V},$$

$$n_s = 1 - 6\epsilon_V + 2\eta_V\,, \qquad r = 16\epsilon_V\,,$$

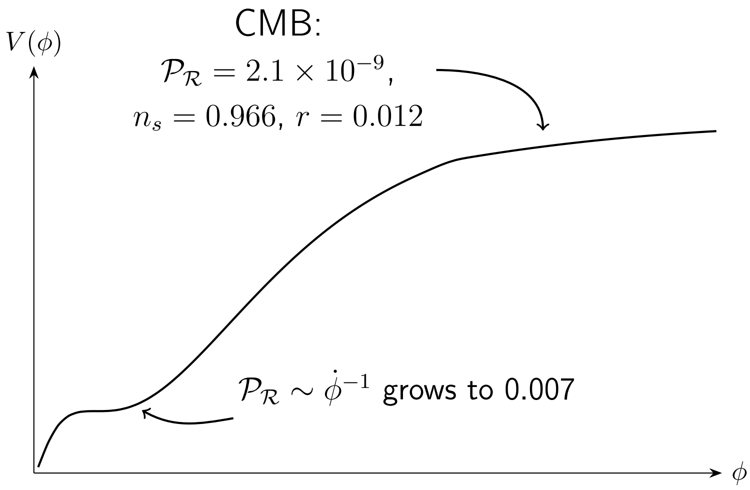
$$\epsilon_V = \frac{M_P^2}{2}\left(\frac{V'}{V}\right)^2, \quad \eta_V = M_P^2 \frac{V''}{V}\,.$$

Observations (Planck):

$$k_* = 0.05\text{Mpc}^{-1}\,, \quad A_s \approx 2.1 \times 10^{-9}\,,$$

$$n_s \approx 0.96\,, \qquad r \lesssim 0.08$$

Our example model



Must go beyond linear perturbations

Coarse-grain perturbations over super-Hubble scales

Gradient expansion: to leading order, coarse-grained perturbations follow locally (non-linear) FLRW equations [Class.Q.Grav.9,1943(1992)]

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Gradient expansion: to leading order, coarse-grained perturbations follow locally (non-linear) FLRW equations [Class.Q.Grav.9,1943(1992)]

ΔN formalism: from FLRW variables to perturbation variables [astro-ph/9507001]

- Change in e-folds of expansion $\Delta N = \Delta \ln a =$ curvature perturbation $\mathcal{R}$

Stretching perturbations give stochastic kicks

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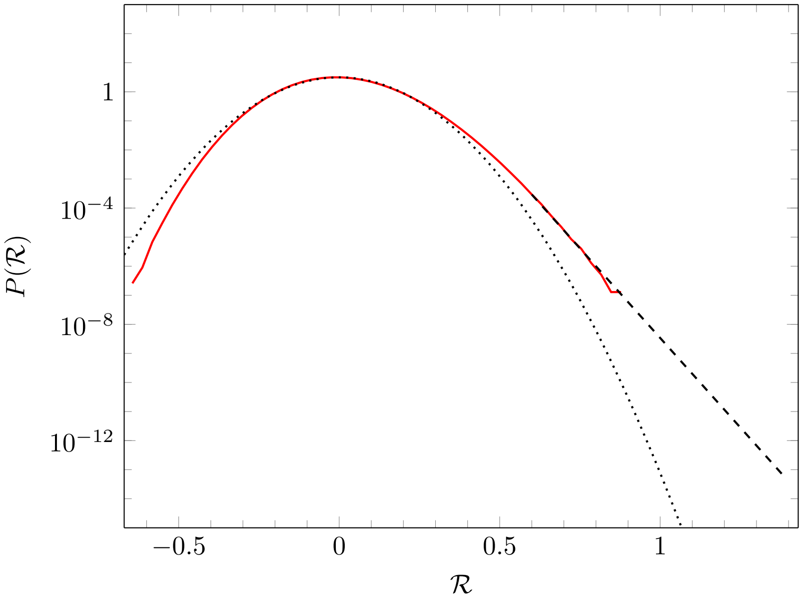
Stochastic evolution of local coarse-grained field
[Lect.Notes Phys.246,107(1986)]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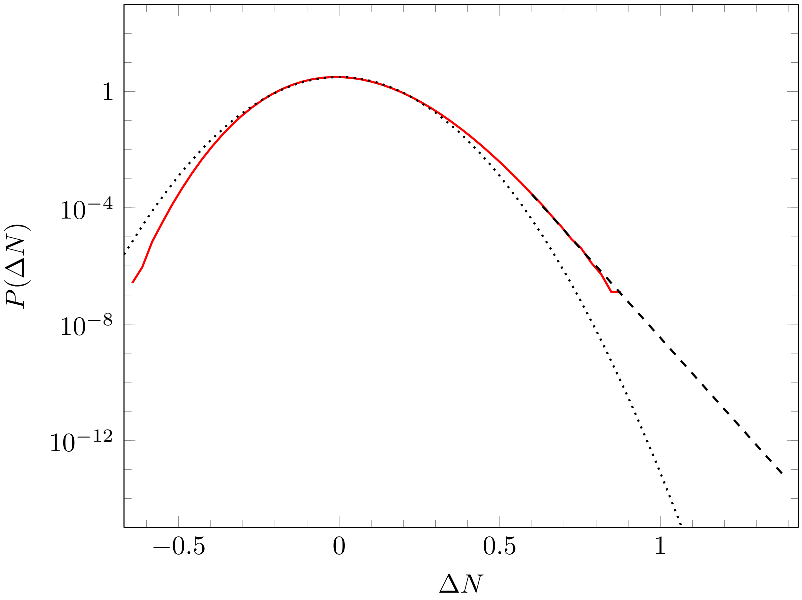

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[1309.4201, 1405.7023, 2011.03014]

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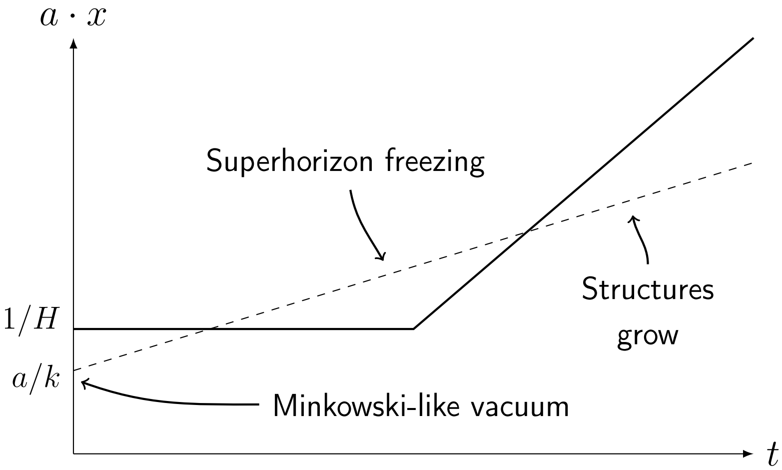
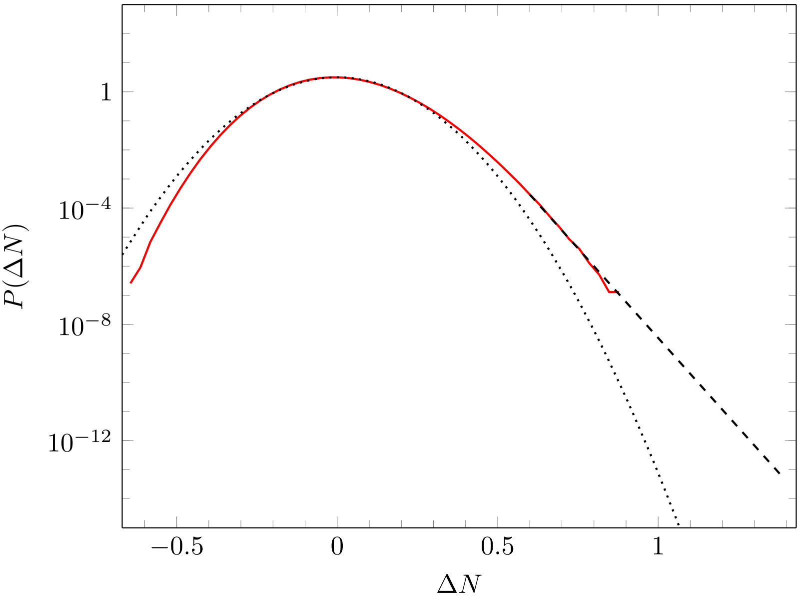
Perturbation collapses to black hole if it exceeds threshold
`[1309.4201, 1405.7023, 2011.03014]`

BH mass = all the mass inside one Hubble radius when the scale re-enters

Evolution of length scales

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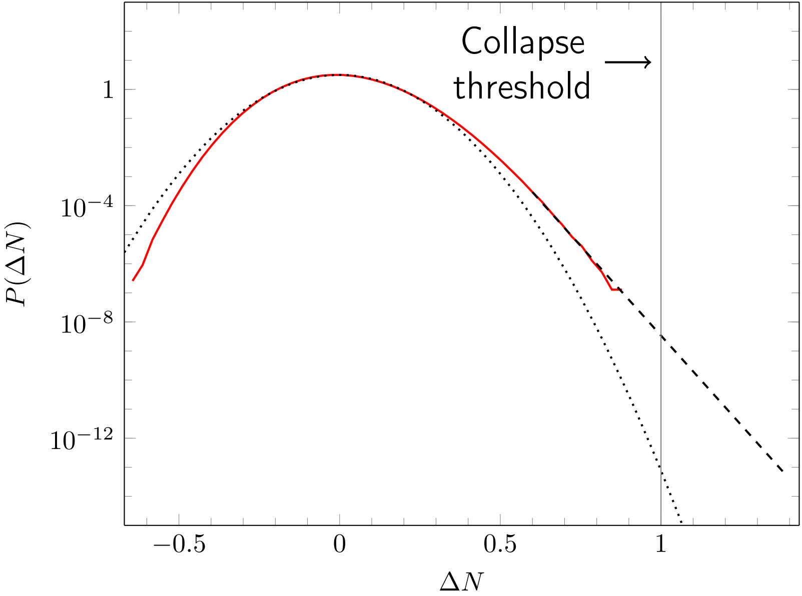
$P(\Delta N)$
−0.5
0
0.5
1
ΔN


Collapse threshold
1
10^{-4}
10^{-8}
10^{-12}
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−0.5
0
0.5
1
ΔN

Dividing the field

Divide inflaton field ϕ into coarse-grained and short wavelength perturbations:

$$\phi = \underbrace{\int_{k<k_\mathrm{c}} \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}}\, \phi_{\vec{k}}\, e^{-i\vec{k}\cdot\vec{x}}}_{\bar{\phi}} + \underbrace{\int_{k>k_\mathrm{c}} \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}}\, \phi_{\vec{k}}\, e^{-i\vec{k}\cdot\vec{x}}}_{\delta\phi}$$

with coarse-graining scale $k_\mathrm{c} = \sigma a H$, $\sigma < 1$

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Define coarse-grained field momentum:

$$\bar{\pi} = \int_{k<k_\mathrm{c}} \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}}\, \frac{\partial}{\partial N} \phi_{\vec{k}}\, e^{-i\vec{k}\cdot\vec{x}}$$

Coarse-graining induces noise

Time derivatives:

$$\bar{\phi}' = \bar{\pi} + \xi_\phi$$

$$\bar{\pi}' = \int_{k<k_\mathrm{c}} \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \frac{\partial^2}{\partial N^2} \phi_{\vec{k}}\, e^{-i\vec{k}\cdot\vec{x}} + \xi_\pi$$

ξ_ϕ, ξ_π are noise from drifting Fourier-modes (random due to quantum initial conditions)

Field equations become stochastic

Full scalar field equation:

$\partial^\mu \partial_\mu \phi - V'(\phi) = 0$

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Full scalar field equation:

$$\phi'' + \left(3 + \frac{H'}{H}\right)\phi' - \frac{1}{a^2H^2}\nabla^2\phi + \frac{V'(\phi)}{H^2} = 0$$

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Full scalar field equation:

$$\int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\,\frac{\partial^2}{\partial N^2}\phi_{\vec{k}}\,e^{-i\vec{k}\cdot\vec{x}} + \Big(3+\frac{H'}{H}\Big)(\bar{\pi}+\delta\pi)$$

$$-\frac{1}{a^2H^2}\nabla^2\bar{\phi} - \frac{1}{a^2H^2}\nabla^2\delta\phi$$

$$+\frac{1}{H^2}\Big(V'(\bar{\phi}) + \frac{1}{2}V''(\bar{\phi})\delta\phi + \frac{1}{6}V'''(\bar{\phi})\delta\phi^2 + \dots\Big)$$

$$= 0$$

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$$-\frac{1}{a^2H^2}\nabla^2\bar{\phi} \xrightarrow{} 0 - \frac{1}{a^2H^2}\nabla^2\delta\phi$$

$$+\frac{1}{H^2}\left(V'(\bar{\phi}) + \frac{1}{2}V''(\bar{\phi})\delta\phi + \frac{1}{6}V'''(\bar{\phi})\delta\phi^2 + \ldots\right) \xrightarrow{} 0$$

$$= 0$$

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$$\delta\phi''_{\vec{k}} + \left(3 + \frac{H'}{H}\right)\delta\phi'_{\vec{k}} + \left(\frac{k^2}{a^2H^2} + \frac{1}{H^2}V''(\bar{\phi})\right)\delta\phi_{\vec{k}} = 0$$

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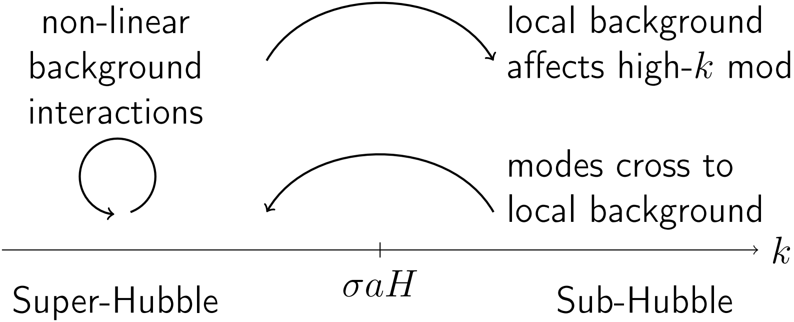
$$3H^2 = \frac{1}{2}\bar{\pi}^2 + V(\bar{\phi})$$

Non-linear interactions included



Perturbations start in Bunch-Davies vacuum

Perturbation initial conditions are

$$\delta\phi_{\vec{k}} = \frac{1}{a\sqrt{2k}}\,, \qquad \delta\phi'_{\vec{k}} = -\Big(1 + i\frac{k}{aH}\Big)\delta\phi_{\vec{k}}$$

We follow modes from deep within the Hubble radius to coarse-graining scale to get the kicks

Discrete time steps give finite kicks

Free quantum scalar field: Gaussian statistics, white noise,

$$\left\langle \xi_\phi^2 \right\rangle = \left\langle (\Delta\bar{\phi})^2 \right\rangle = \mathrm{d}N \frac{k^3}{2\pi^2}\left(1 + \frac{H'}{H}\right) \left|\delta\phi_{\vec{k}}\right|^2$$

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Squeezed state: ξ_ϕ and ξ_π are highly correlated, so that $\Delta\bar{\pi} = \frac{\delta\phi'_{\vec{k}}}{\delta\phi_{\vec{k}}}\Delta\bar{\phi}$

Kicks are turned off when target scale reached

We are interested in PBHs with $M_{\rm PBH} = 10^{-14} M_{\odot}$, corresponding to $k_{\rm PBH} = 10^{13}$ Mpc^{-1}

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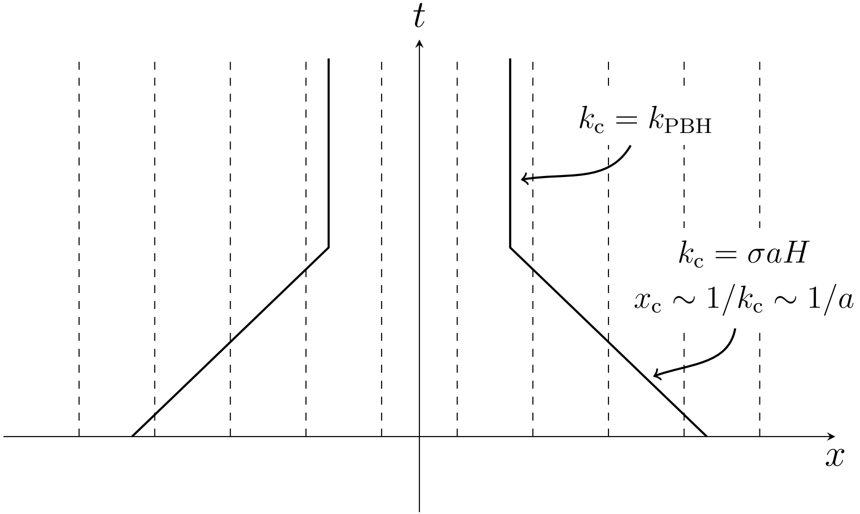
Continue evolution to a fixed field value and store $\Delta N = \mathcal{R}$

Evolution of patch size



Algorithm 1: Evolution for each run

Set initial values for N, $\bar{\phi}$, $\bar{\pi}$. Set $k_{\text{next}} = k_*$. Set current kick coefficient to zero.

while $\bar{\phi} > \bar{\phi}_{\text{f}}$ **do**

  Evolve N, $\bar{\phi}$, $\bar{\pi}$.

  for *all modes k in the simulation* **do**

    if $k > \sigma a H$ **then**

      Evolve $\delta\phi_{\vec{k}}$, $\delta\phi'_{\vec{k}}$.

    else

      Evolve $\delta\phi_{\vec{k}}$, $\delta\phi'_{\vec{k}}$ to $k = \sigma a H$. Set the current kick coefficient from $\delta\phi_{\vec{k}}$, $\delta\phi'_{\vec{k}}$. Remove mode k from the simulation.

  if $k_{\text{next}} \leq k_{\text{PBH}}$ **then**

    if $k_{\text{next}} \leq \alpha a H$ **then**

      Add mode $k = k_{\text{next}}$ to the simulation. Set initial values for $\delta\phi_{\vec{k}}$, $\delta\phi'_{\vec{k}}$. Evolve $\delta\phi_{\vec{k}}$, $\delta\phi'_{\vec{k}}$ from $k = \alpha a H$. Set $k_{\text{next}} = e^{1/32} k_{\text{next}}$.

  else

    if $k_{\text{next}} \leq \sigma a H$ **then**

      Set the current kick coefficient to zero.

  Add stochastic kick to $\bar{\phi}$, $\bar{\pi}$ using the current kick coefficient.

Want tiny initial PBH fraction

PBH fraction today:

$$\Omega_{\mathsf{PBH}} \approx 9 \times 10^7 \gamma^{\frac{1}{2}} \beta \left(\frac{M}{M_\odot} \right)^{-\frac{1}{2}} \sim 0.3$$

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Need initial fraction $\beta \sim 10^{-16}$

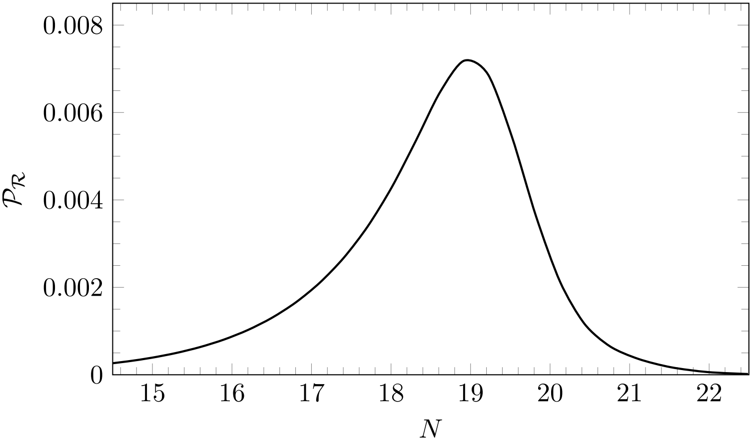
Model fitted by Gaussian approximation

With Gaussian statistics:

$$\sigma_{\mathcal{R}}^2 \equiv \int^{k_{\rm PBH}} \mathrm{d}(\ln k) \mathcal{P}_{\mathcal{R}}(k)$$

$$\beta = 2 \int_{\mathcal{R}_c}^{\infty} \mathrm{d}\mathcal{R} \frac{1}{\sqrt{2\pi}\sigma_{\mathcal{R}}} e^{-\frac{\mathcal{R}^2}{2\sigma_{\mathcal{R}}^2}} \approx \frac{\sqrt{2}\sigma_{\mathcal{R}}}{\sqrt{\pi}\mathcal{R}_c} e^{-\frac{\mathcal{R}_c^2}{2\sigma_{\mathcal{R}}^2}}$$

Model fitted by Gaussian approximation



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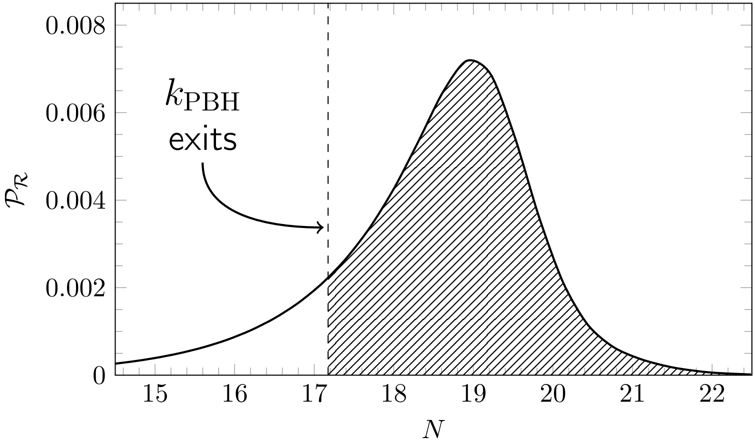

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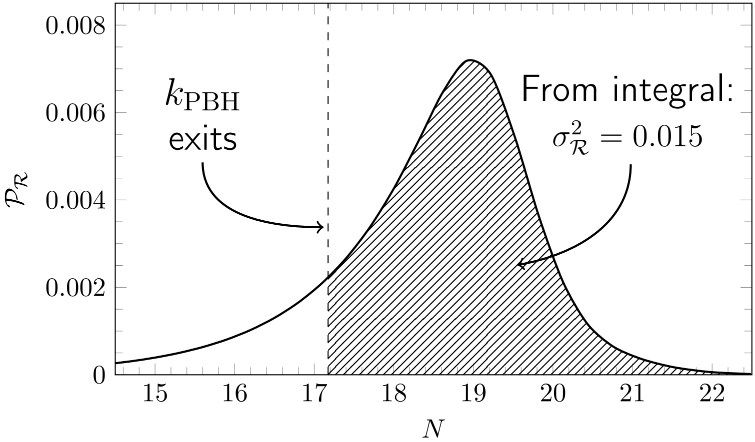

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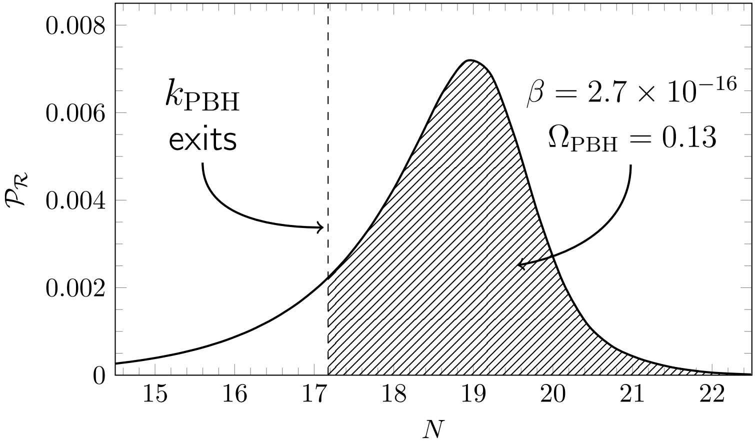

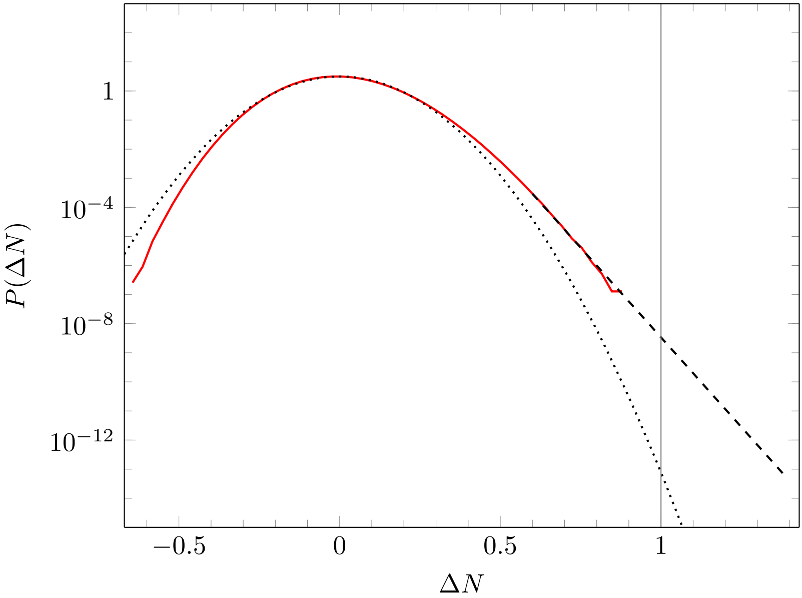
$P(\Delta N)$
1
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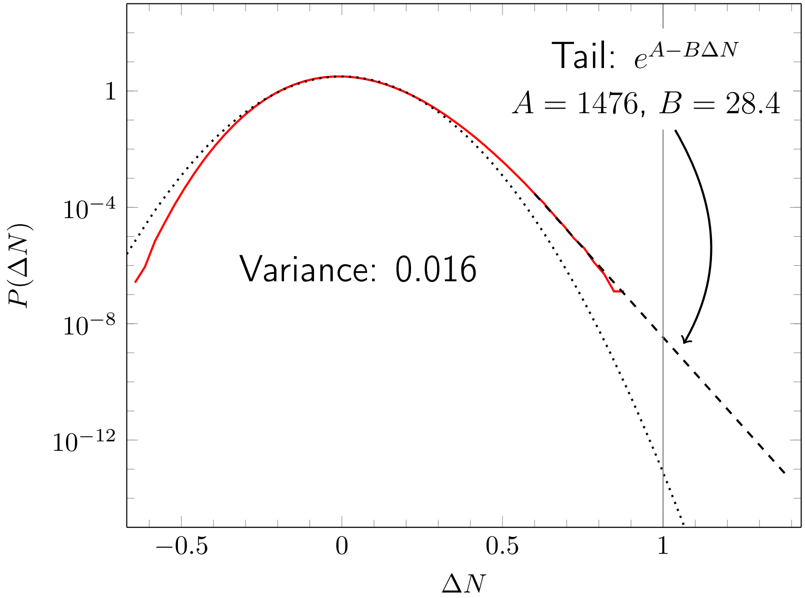
0
0.5
1
ΔN


Tail: $e^{A-B\Delta N}$
$A = 1476$, $B = 28.4$
Variance: 0.016
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10^{-4}
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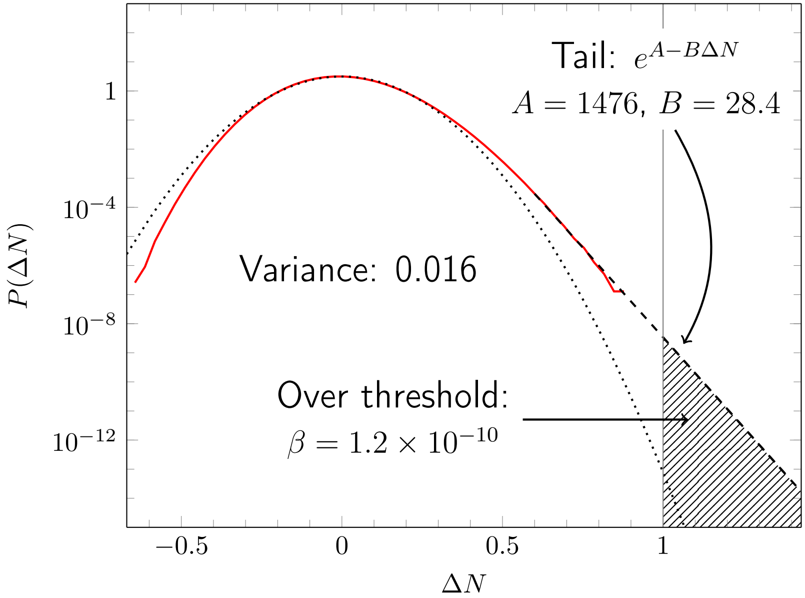
Tail: $e^{A-B\Delta N}$
$A = 1476$, $B = 28.4$
Variance: 0.016
Over threshold:
$\beta = 1.2 \times 10^{-10}$
$P(\Delta N)$
1
10^{-4}
10^{-8}
10^{-12}
−0.5
0
0.5
1
ΔN

...true abundance much higher

Numerics: exponential tail, with

$$\beta = 1.2 \times 10^{-10}\,, \quad \Omega_{\rm PBH} = 5.4 \times 10^{4}$$

Larger than Gaussian result by factor 10^5!

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Other sources of error: uncertainty in $\mathcal{R}_c$, window functions, different Gaussian computation schemes, ...

Future directions

More statistics

More models

Full mass spectrum

Correlations between different scales

Conclusions

Inflation produces cosmological perturbations

Strongest perturbations collapse to black holes

Non-Gaussian tail of probablity distribution
important for black hole statistics

Stochastic inflation allows us to probe this

Thank you!

[2012.06551]

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Coarse-graining parameter $\sigma < 1$ is a free parameter

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Want to make a physically well-motivated choice

- Want a lot of non-linear interactions: large σ
- Want kicks to be classical: small σ

Demanding high squeezing sets σ

Classicality measured by squeezing of quantum state

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- Also, ξ_ϕ and ξ_π correlated

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$$\cosh(2r_k) = a^3\big(\tfrac{k}{a}|\delta\phi_k|^2 + \tfrac{a}{k}H^2|\delta\phi_k'|^2\big)$$

Our choice: $\sigma = 0.01$ ensures $\cosh(2r_k) > 100$ for all modes when they exit k_{c}

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$\delta\phi$ and thus kicks solved in spatially flat gauge

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Tests and theory: no significant difference [1905.06300]

Model details

$$V = \frac{\lambda(h)}{4} F(h)^4$$

$$F(h) = \frac{Ah}{\sqrt{1+B\xi(h-C)^2}}, \qquad \frac{\mathrm{d}h}{\mathrm{d}\chi} = \frac{1+\xi h^2}{\sqrt{1+\xi h^2+6\xi^2 h^2}}$$

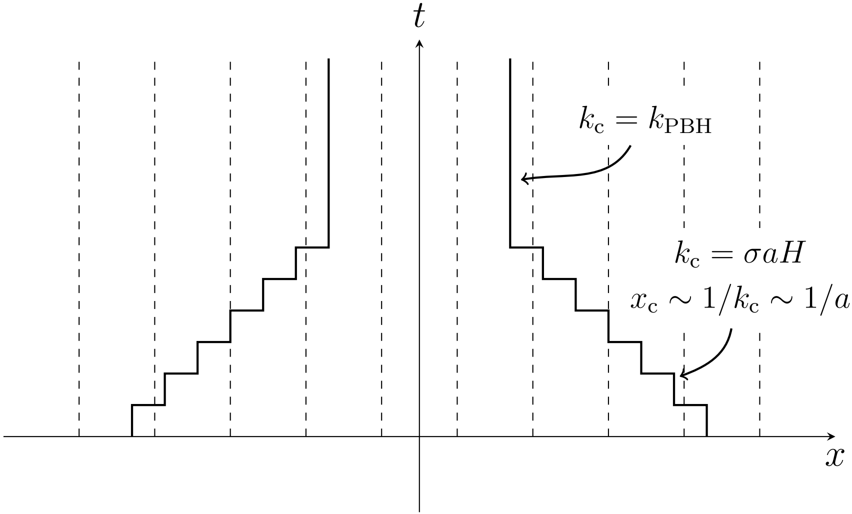
$$\xi = 38.8$$

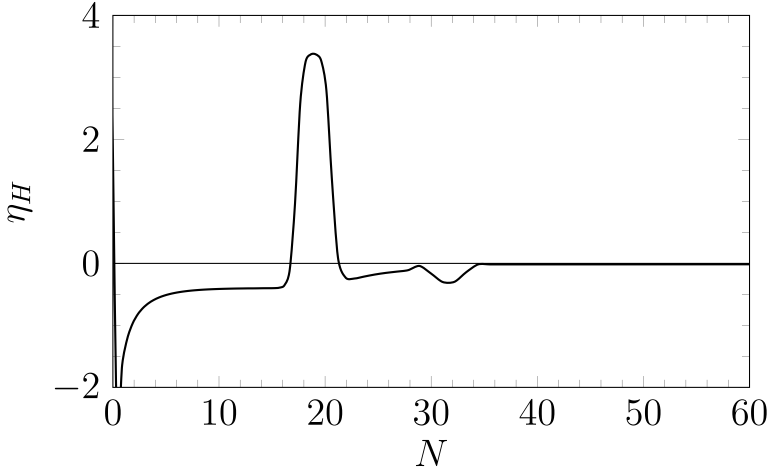
$n_s = 0.966$, $r = 0.012$, $A_s = 2.1 \times 10^{-9}$

USR between 17.2 and 20.8 e-folds

[1810.12608]

Evolution of patch size





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