A numerical approach to stochastic inflation and primordial black holes

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Based on 2012.06551, in collaboration with D. Figueroa, S. Raatikainen, S. Räsänen

Concepts

Cosmic inflation

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Cosmic inflation

Cosmological perturbations

Concepts

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Cosmological perturbations

Primordial black holes

Concepts

Stochastic inflation

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Stochastic inflation

- Includes non-linear effects

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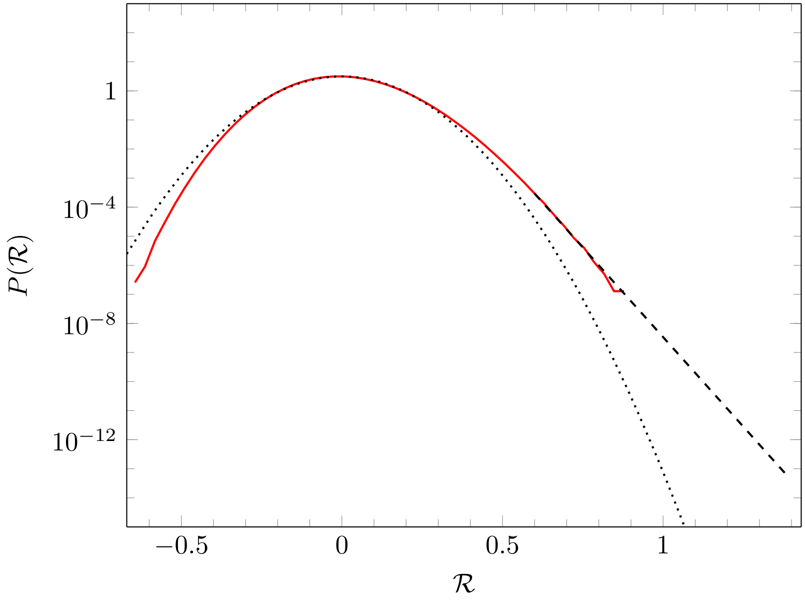
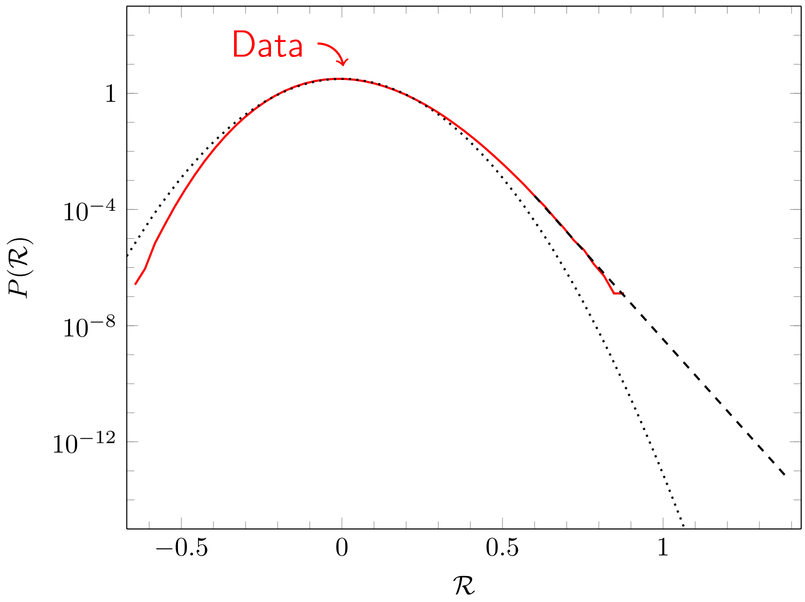
Stochastic inflation

- Includes non-linear effects
- Numerical method: even more non-linearities



$P(\mathcal{R})$
1
10^{-4}
10^{-8}
10^{-12}
−0.5
0
0.5
1
$\mathcal{R}$


Data
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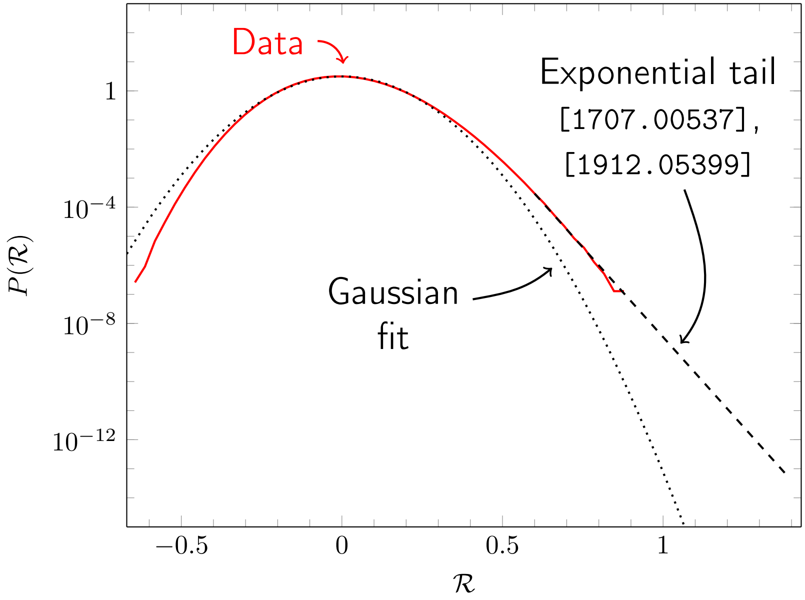
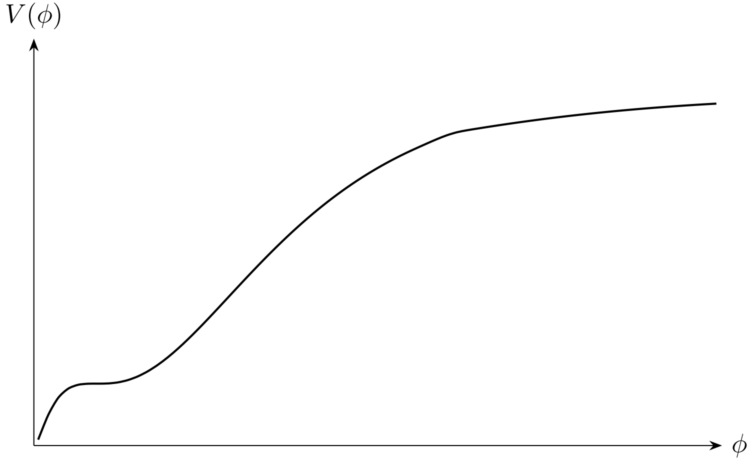


Data
Exponential tail
[1707.00537],
[1912.05399]
Gaussian fit
1
10−4
10−8
10−12
P(R)
−0.5
0
0.5
1
R

Model of inflation fits CMB



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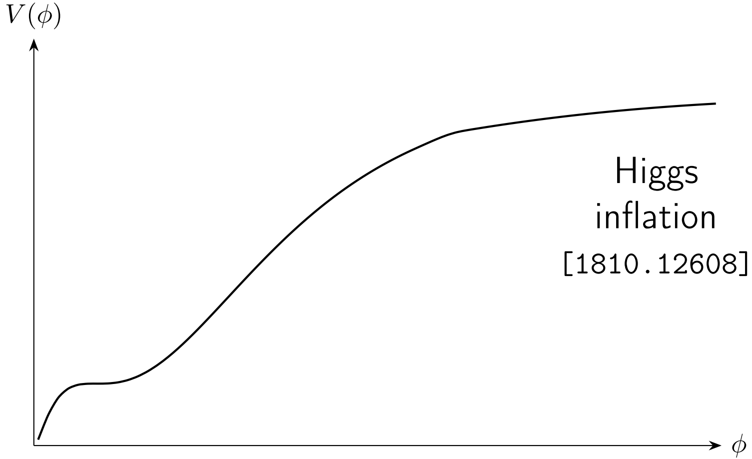

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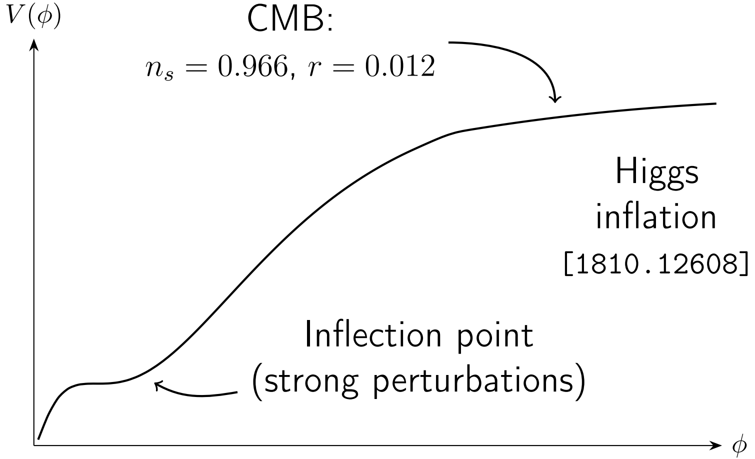

Perturbations depend on scale

Origin of perturbations: fluctuations of quantum vacuum

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Space expands and perturbations get stretched

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Perturbations (eventually) become classical and freeze after crossing Hubble horizon

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Origin of perturbations: fluctuations of quantum vacuum

Space expands and perturbations get stretched

Perturbations (eventually) become classical and freeze after crossing Hubble horizon

Strong perturbations from ultra-slow-roll inflation

Must go beyond linear perturbations

Coarse-grain perturbations over super-Hubble scales

Gradient expansion: to leading order, coarse-grained perturbations follow locally (non-linear) FLRW equations `[Class.Q.Grav.9,1943(1992)]`

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ΔN formalism: from FLRW variables to perturbation variables [astro-ph/9507001]

- Change in e-folds of expansion $\Delta N =$ curvature perturbation $\mathcal{R}$

Stretching perturbations give stochastic kicks

When perturbations of a certain scale stretch to the coarse-graining scale, they get coarse-grained

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Result: 'kicks' to coarse-grained field.
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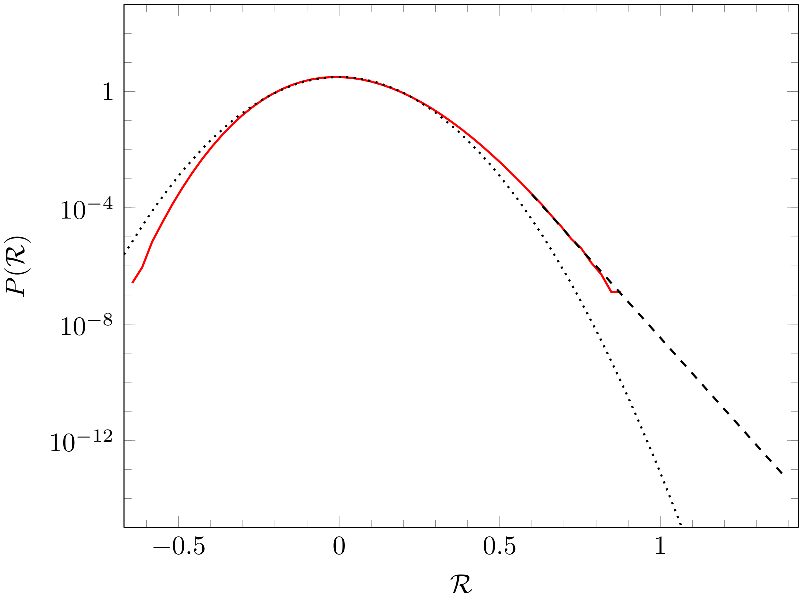
Stochastic evolution of local coarse-grained field
[Lect.Notes Phys.246,107(1986)]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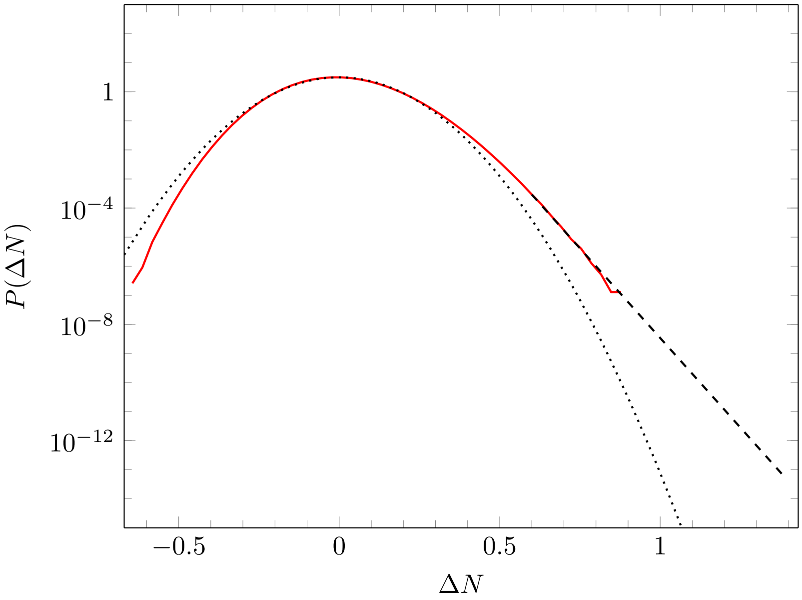

1
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ΔN

PBHs form from strong perturbations

During radiation domination, perturbations re-enter Hubble radius

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Perturbation collapses to black hole if it exceeds threshold
`[1309.4201, 1405.7023, 2011.03014]`

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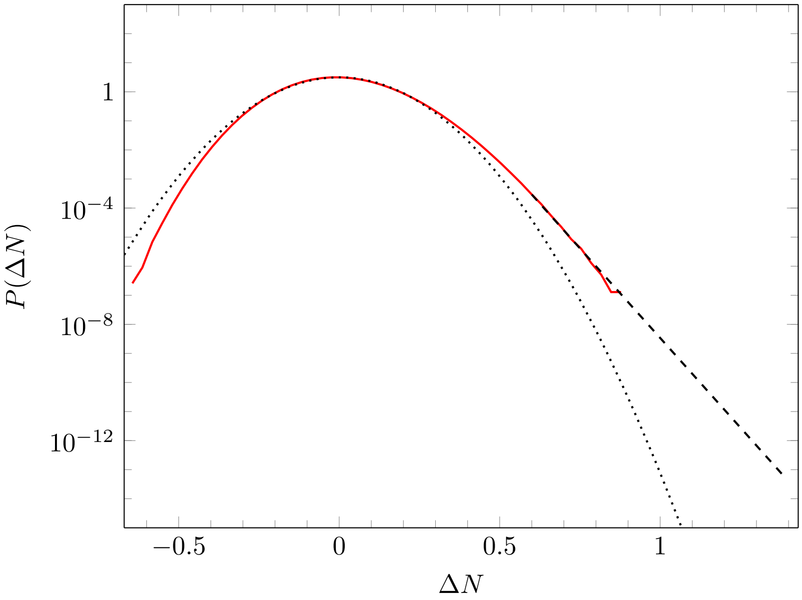
Perturbation collapses to black hole if it exceeds threshold
`[1309.4201, 1405.7023, 2011.03014]`

BH mass = all the mass inside one Hubble radius when the scale re-enters



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1
10^{-4}
10^{-8}
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-0.5
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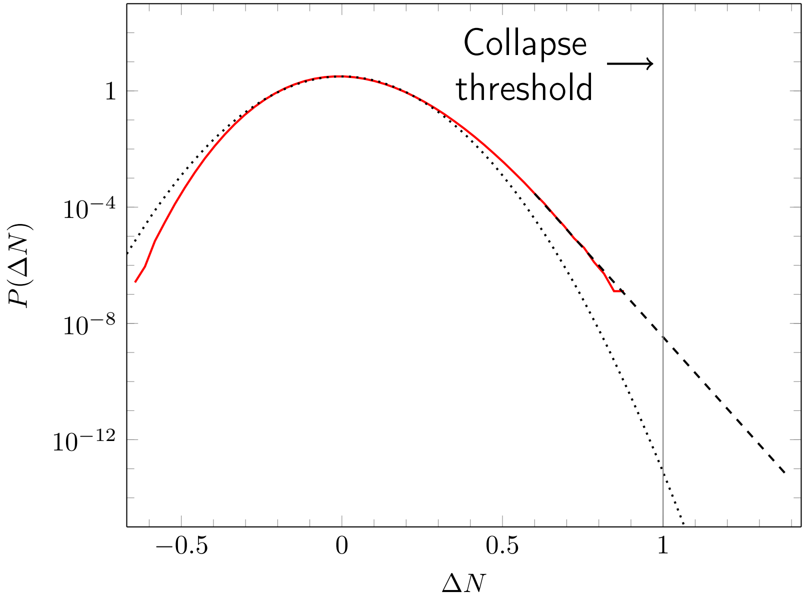
1
ΔN


Collapse threshold
1
10^{-4}
10^{-8}
10^{-12}
$P(\Delta N)$
−0.5
0
0.5
1
ΔN

Dividing the field

Divide inflaton field ϕ into coarse-grained and short wavelength perturbations:

$$\phi = \underbrace{\int_{k<k_\mathrm{c}} \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}}\, \phi_{\vec{k}}\, e^{-i\vec{k}\cdot\vec{x}}}_{\bar{\phi}} + \underbrace{\int_{k>k_\mathrm{c}} \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}}\, \phi_{\vec{k}}\, e^{-i\vec{k}\cdot\vec{x}}}_{\delta\phi}$$

with coarse-graining scale $k_\mathrm{c} = \sigma a H$, $\sigma < 1$

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Define coarse-grained field momentum:

$$\bar{\pi} = \int_{k<k_\mathrm{c}} \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}}\, \frac{\partial}{\partial N} \phi_{\vec{k}}\, e^{-i\vec{k}\cdot\vec{x}}$$

Coarse-graining induces noise

Time derivatives:

$$\bar{\phi}' = \bar{\pi} + \xi_\phi$$

$$\bar{\pi}' = \int_{k<k_\mathrm{c}} \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \frac{\partial^2}{\partial N^2} \phi_{\vec{k}} \, e^{-i\vec{k}\cdot\vec{x}} + \xi_\pi$$

ξ_ϕ, ξ_π are noise from drifting Fourier-modes (random due to quantum initial conditions)

Field equations become stochastic

Full scalar field equation:

$\partial^\mu \partial_\mu \phi - V'(\phi) = 0$

Field equations become stochastic

Full scalar field equation:

$$\phi'' + \left(3 + \frac{H'}{H}\right)\phi' - \frac{1}{a^2H^2}\nabla^2\phi + \frac{V'(\phi)}{H^2} = 0$$

Field equations become stochastic

Full scalar field equation:

$$\int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\, \frac{\partial^2}{\partial N^2}\phi_{\vec{k}}\, e^{-i\vec{k}\cdot\vec{x}} + \left(3 + \frac{H'}{H}\right)(\bar{\pi} + \delta\pi)$$

$$-\frac{1}{a^2H^2}\nabla^2\bar{\phi} - \frac{1}{a^2H^2}\nabla^2\delta\phi$$

$$+\frac{1}{H^2}\left(V'(\bar{\phi}) + \frac{1}{2}V''(\bar{\phi})\delta\phi + \frac{1}{6}V'''(\bar{\phi})\delta\phi^2 + \dots\right)$$

$$= 0$$

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$$\bar{\phi}' = \bar{\pi} + \xi_\phi$$

$$\delta\phi''_{\vec{k}} + \left(3 + \tfrac{H'}{H}\right)\delta\phi'_{\vec{k}} + \left(\tfrac{k^2}{a^2H^2} + \tfrac{1}{H^2}V''(\bar{\phi})\right)\delta\phi_{\vec{k}} = 0$$

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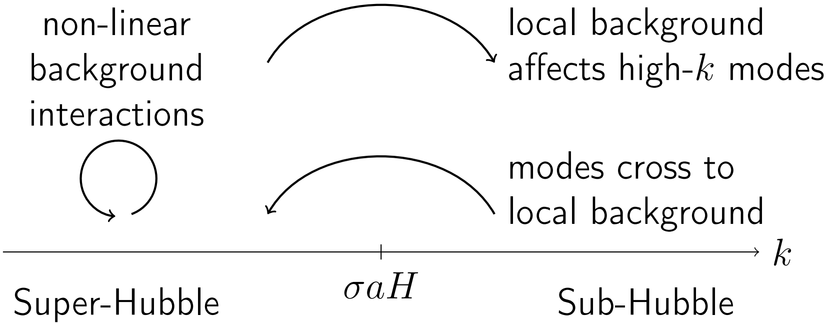
$$3HH' + \left(3 + \bar{\pi}^2\right)H^2 = V(\bar{\phi})$$

Non-linear interactions included



Perturbations start in Bunch-Davies vacuum

Perturbation initial conditions are

$$\delta\phi_{\vec{k}} = \frac{1}{a\sqrt{2k}}\,, \qquad \delta\phi'_{\vec{k}} = -\left(1 + i\frac{k}{aH}\right)\delta\phi_{\vec{k}}$$

We follow modes from deep within the Hubble radius to coarse-graining scale to get the kicks

Discrete time steps give finite kicks

Free quantum scalar field: Gaussian statistics, white noise,

$$\left\langle \xi_\phi^2 \right\rangle = \left\langle (\Delta\bar{\phi})^2 \right\rangle = \mathrm{d}N \frac{k^3}{2\pi^2}\left(1 + \frac{H'}{H}\right) \left|\delta\phi_{\vec{k}}\right|^2$$

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Squeezed state: ξ_ϕ and ξ_π are highly correlated, so that $\Delta\bar{\pi} = \frac{\delta\phi'_{\vec{k}}}{\delta\phi_{\vec{k}}}\Delta\bar{\phi}$

Kicks are turned off when target scale reached

We are interested in PBHs with $M_{\rm PBH} = 10^{-14} M_\odot$, corresponding to $k_{\rm PBH} = 10^{13}$ Mpc^{-1}

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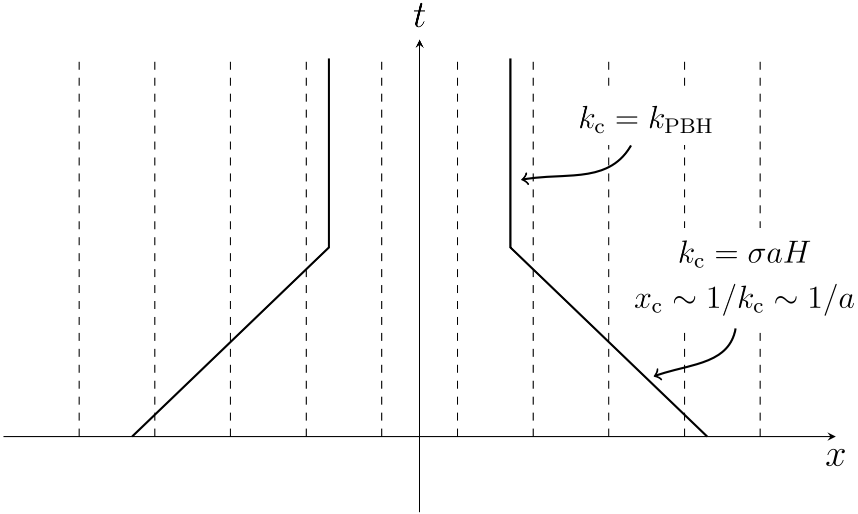
Continue evolution to a fixed field value and store $\Delta N = \mathcal{R}$

Evolution of patch size



Algorithm 1: Evolution for each run

```
Set initial values for N, H, φ̄, π̄.
while φ̄ > φ̄_f do
    Evolve H, φ̄, π̄ one time step (without
      kicks).
    for k ∈ {k_1, k_2, ...} do
        if k = αaH then
            Set initial values for δφ_k⃗, δφ'_k⃗.
        if σaH < k < αaH then
            Evolve δφ_k⃗, δφ'_k⃗ one time step.
    Add stochastic kick to φ̄, π̄ from the
      most recent mode with k < σaH,
      unless k > k_PBH.
```

Want tiny initial PBH fraction

Scale $M_{\rm PBH} = 10^{-14} M_\odot$, $k_{\rm PBH} = 10^{13}$ Mpc^{-1} chosen so that USR ends when $k_{\rm PBH}$ gets coarse-grained

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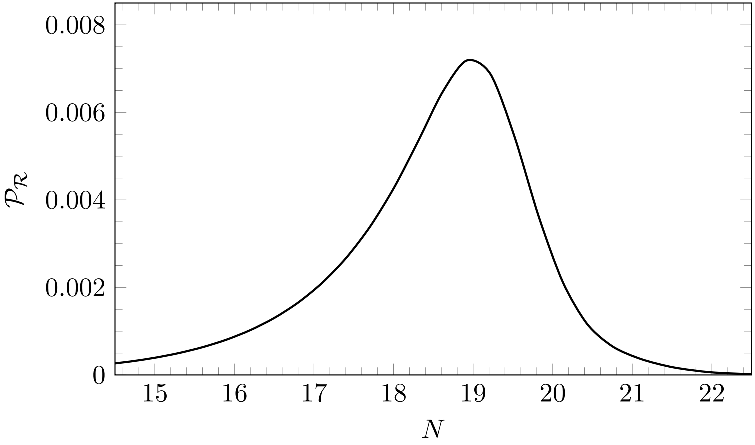
Gaussian statistics:

$\sigma_{\mathcal{R}}^2 = \int^{k_{\rm PBH}} \mathrm{d}(\ln k) \mathcal{P}_{\mathcal{R}}(k)$

$\beta = 2 \int_{\mathcal{R}_c}^{\infty} \mathrm{d}\mathcal{R} \frac{1}{\sqrt{2\pi}\sigma_{\mathcal{R}}} e^{-\frac{\mathcal{R}^2}{2\sigma_{\mathcal{R}}^2}} \approx \frac{\sqrt{2}\sigma_{\mathcal{R}}}{\sqrt{\pi}\mathcal{R}_c} e^{-\frac{\mathcal{R}_c^2}{2\sigma_{\mathcal{R}}^2}}$

Model fitted by Gaussian approximation...



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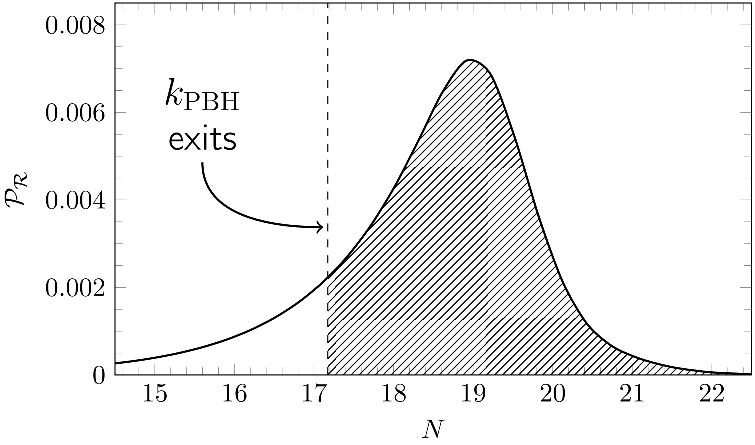

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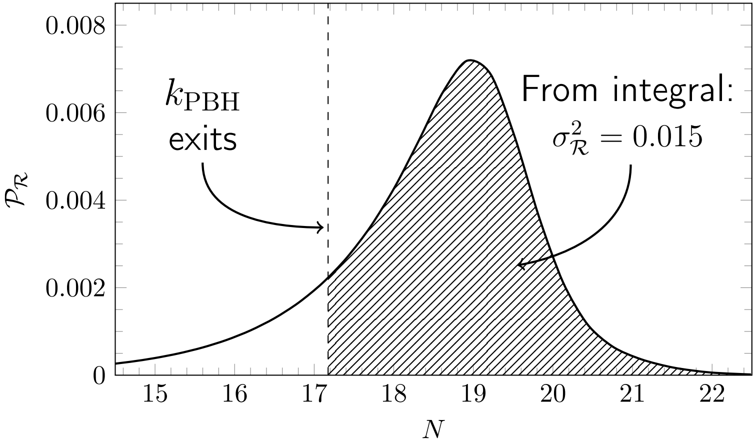

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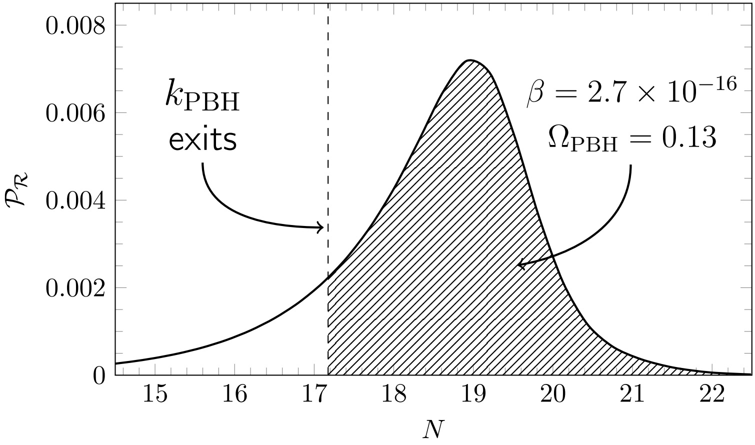

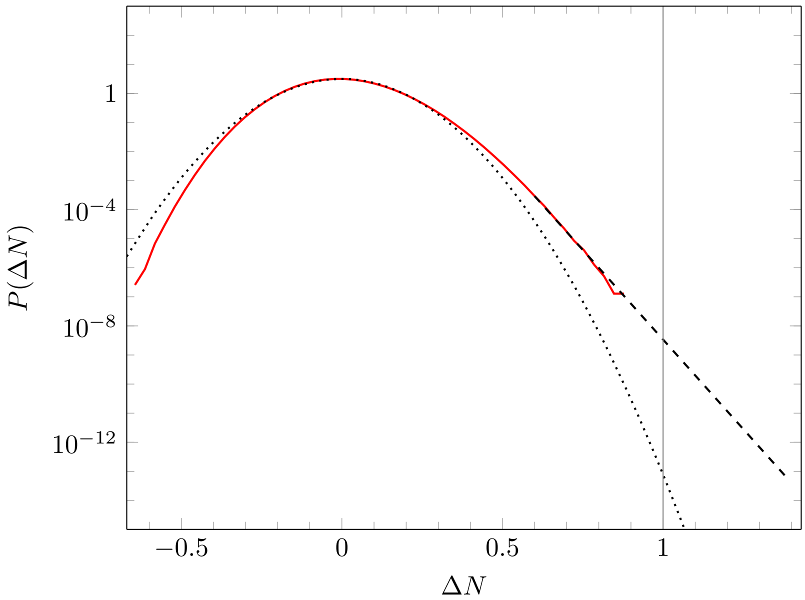
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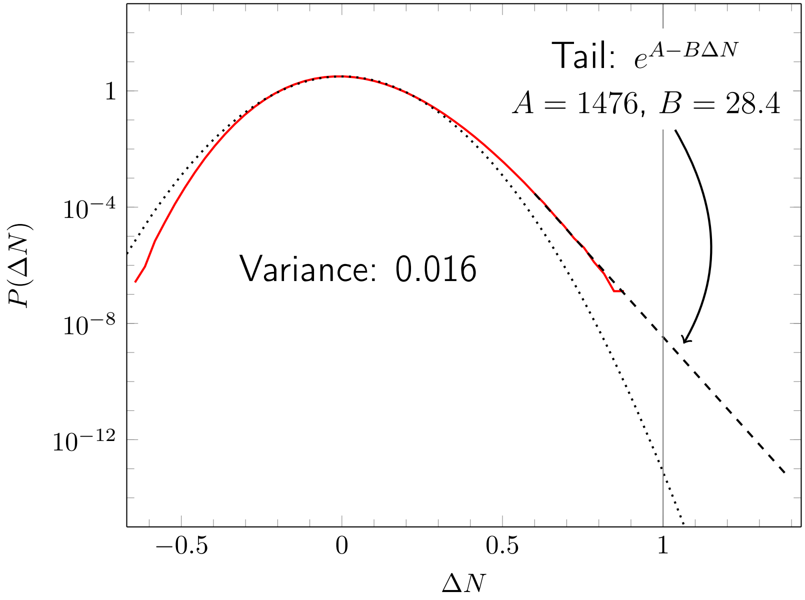

Tail: $e^{A-B\Delta N}$
$A = 1476$, $B = 28.4$
Variance: 0.016
$P(\Delta N)$
1
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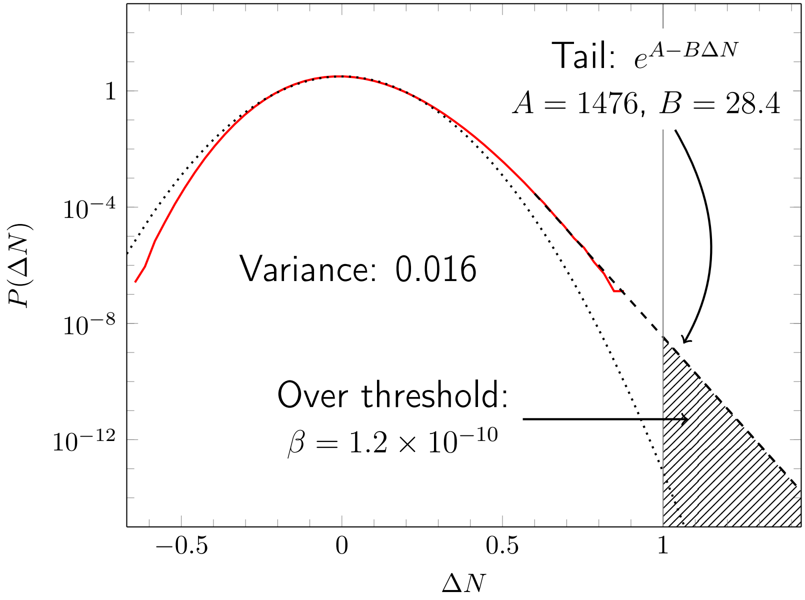

Tail: $e^{A-B\Delta N}$
$A = 1476,\ B = 28.4$
Variance: 0.016
Over threshold:
$\beta = 1.2 \times 10^{-10}$
$P(\Delta N)$
1
10^{-4}
10^{-8}
10^{-12}
-0.5
0
0.5
1
ΔN

...true abundance much higher

Numerics: exponential tail, with

$$\beta = 1.2 \times 10^{-10}\,, \quad \Omega_{\rm PBH} = 5.4 \times 10^{4}$$

Larger than Gaussian result by factor 10^5!

What about σ?

Coarse-graining parameter $\sigma < 1$ is a free parameter

- Results may depend on it

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Coarse-graining parameter $\sigma < 1$ is a free parameter

- Results may depend on it

Want to make a physically well-motivated choice

- Want a lot of non-linear interactions: large σ
- Want kicks to be classical: small σ

Demanding high squeezing sets σ

Classicality measured by squeezing of quantum state

- Squeezed state: phase space probability distribution classial
- Also, ξ_ϕ and ξ_π correlated

Demanding high squeezing sets σ

Classicality measured by squeezing of quantum state

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$$\cosh(2r_k) = a^3\big(\tfrac{k}{a}|\delta\phi_k|^2 + \tfrac{a}{k}H^2|\delta\phi_k'|^2\big)$$

Our choice: $\sigma = 0.01$ ensures $\cosh(2r_k) > 100$ for all modes when they exit k_c

What about gauge issues?

$\delta\phi$ and thus kicks solved in spatially flat gauge

- Easy to solve

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Tests and theory: no significant difference [1905.06300]

Future directions

More statistics

More models

Full mass spectrum

Correlations between different scales

Conclusions

Stochastic inflation captures non-linearities of cosmological perturbations

Crucial for PBH formation

Introduced a numerical recipe to calculate these in a general single-field model

Thank you!

[2012.06551]

Model details

$V = \frac{\lambda(h)}{4} F(h)^4$

$F(h) = \frac{Ah}{\sqrt{1+B\xi(h-C)^2}}, \qquad \frac{\mathrm{d}h}{\mathrm{d}\chi} = \frac{1+\xi h^2}{\sqrt{1+\xi h^2+6\xi^2 h^2}}$

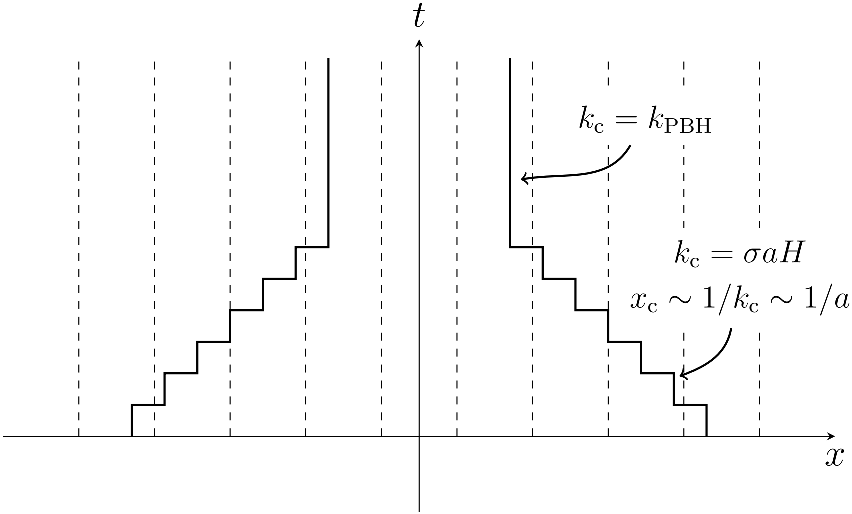
$\xi = 38.8$

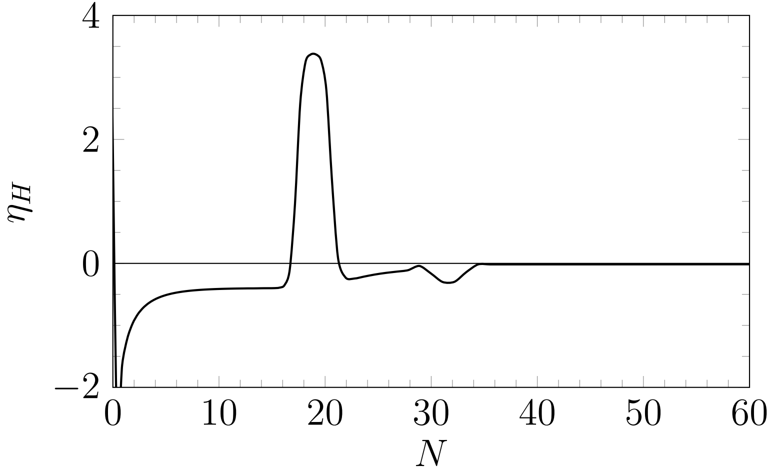
$n_s = 0.966$, $r = 0.012$, $A_s = 2.1 \times 10^{-9}$

USR between 17.2 and 20.8 e-folds

[1810.12608]

Evolution of patch size





4
2
0
−2
η_H
0
10
20
30
40
50
60
N