New developments in stochastic inflation

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Langevin equation

Langevin equation (time variable N = e-folds):

$$\phi_R' = \underbrace{-\frac{V'(\phi_R)}{V(\phi_R)}}_{\equiv\mu(\phi_R)} + \underbrace{\frac{H(\phi_R)}{2\pi}}_{\equiv\sigma(\phi_R)}\xi(N)\,, \qquad \langle\xi(N)\xi(N')\rangle = \delta(N-N')\,.$$

Interpretation for finite time steps?

Itô vs Stratonovich

Itô:

$$\phi(N_+) = \phi(N) + \mu[\phi(N)]\mathrm{d}N + \sigma[\phi(N)]\sqrt{\mathrm{d}N}\,\xi_N$$

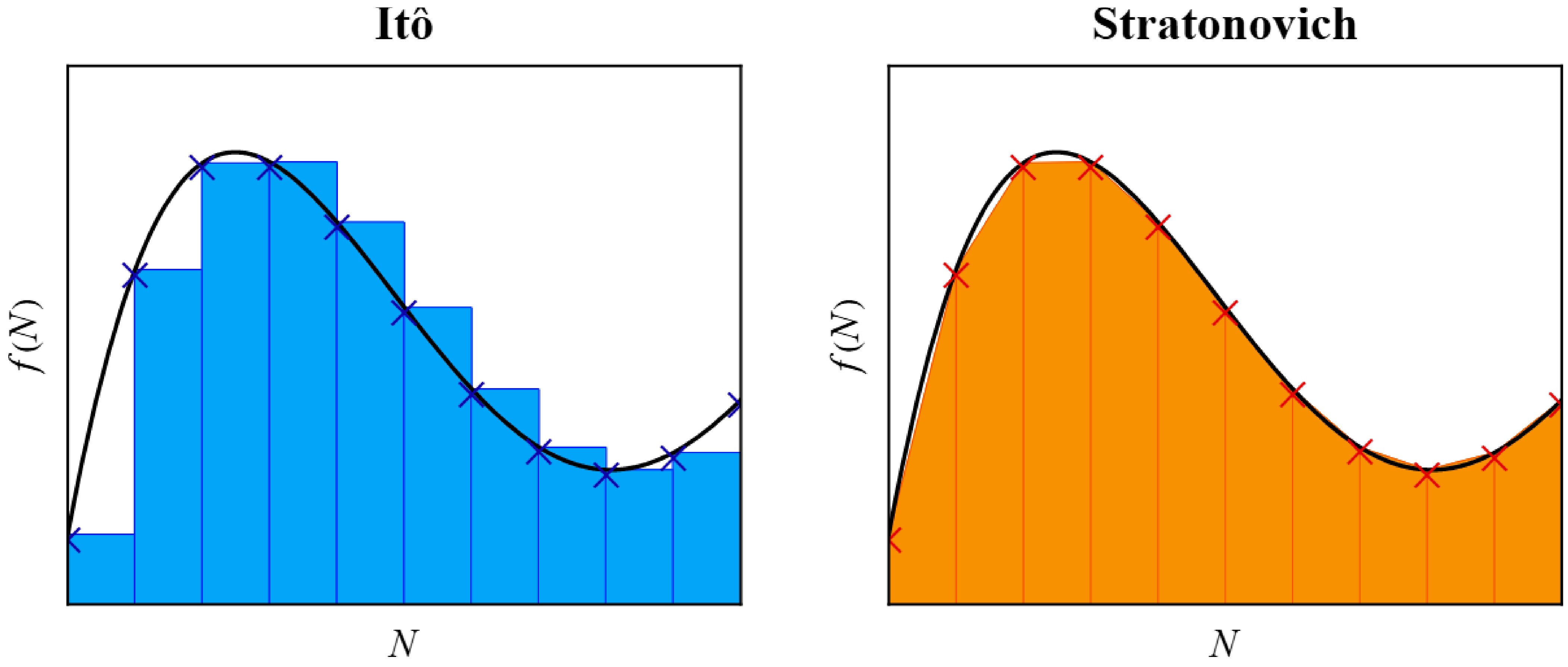
$$N_+ \equiv N + \mathrm{d}N \qquad \langle \xi_N \xi_{N'} \rangle = \delta_{NN'}$$

Stratonovich:

$$\begin{aligned}\phi(N_+) = \phi(N) &+ \frac{1}{2}\{\mu[\phi(N)] + \mu[\phi(N_+)]\}\mathrm{d}N \\ &+ \frac{1}{2}\{\sigma[\phi(N)] + \sigma[\phi(N_+)]\}\sqrt{\mathrm{d}N}\,\xi_N\end{aligned}$$

As integrals

$$\phi'(N) = f[N, \phi(N)] \iff \phi(N) = \int \mathrm{d}N\, f[N, \phi(N)]$$



`Itôfy' Stratonovich

Taylor expand:

$$\mu[\phi(N_+)] = \mu[\phi(N)] + \mu'[\phi(N)]\Delta\phi + \mathcal{O}(\Delta\phi^2)$$

$$\sigma[\phi(N_+)] = \sigma[\phi(N)] + \sigma'[\phi(N)]\Delta\phi + \mathcal{O}(\Delta\phi^2)$$

Iteratively from Stratonovich equation:

$$\Delta\phi = \sigma[\phi(N)]\sqrt{\mathrm{d}N}\xi_N + \mathcal{O}(\mathrm{d}N)$$

`Itôfy' Stratonovich

Stratonovich equation becomes:

$$\phi(N_+) = \phi(N) + \left(\mu[\phi(N)] + \frac{1}{2}\sigma[\phi(N)]\sigma'[\phi(N)]\right)\mathrm{d}N$$
$$+ \sigma[\phi(N)]\sqrt{\mathrm{d}N}\,\xi_N \quad + \quad \mathcal{O}(\mathrm{d}N^{3/2})$$

(used $\xi_N^2 \to 1$)

`Itôfy' Stratonovich

Stratonovich equation becomes:

New drift term!

$$\phi(N_+) = \phi(N) + \left(\mu[\phi(N)] + \boxed{\frac{1}{2}\sigma[\phi(N)]\sigma'[\phi(N)]} \right) \mathrm{d}N + \sigma[\phi(N)]\sqrt{\mathrm{d}N}\,\xi_N \quad + \quad \mathcal{O}(\mathrm{d}N^{3/2})$$

Different physics! (e.g. predictions for cosmological perturbations)

Aside: Fokker-Planck

Itô:

$$\partial_N P(\phi, N) = \partial_\phi \left\{ \frac{1}{2} \partial_\phi [\sigma^2(\phi) P(\phi, N)] - \mu(\phi) P(\phi, N) \right\}$$

Stratonovich:

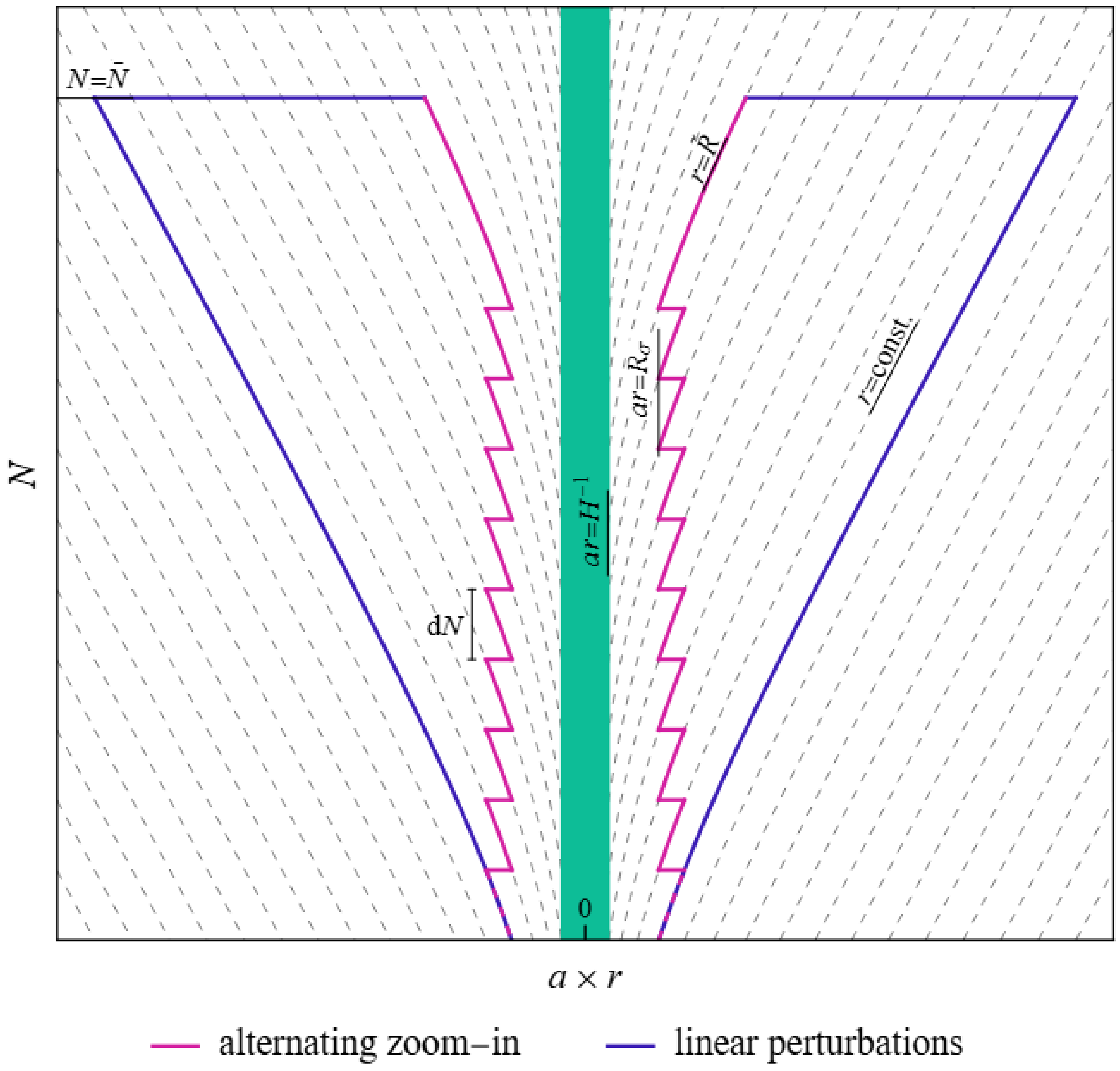
$$\partial_N P(\phi, N) = \partial_\phi \left\{ \frac{1}{2} \sigma(\phi) \partial_\phi [\sigma(\phi) P(\phi, N)] - \mu(\phi) P(\phi, N) \right\}$$

Which to choose?

Itô:
Simple to interpret and implement
Does not respect chain rule*

Stratonovich:
Limit of `coloured noise' (though not unique)
Respects chain rule*

$^{*}\mathrm{d}Y = \frac{\partial Y}{\partial X}\mathrm{d}X$


$N=\bar{N}$
$r=\bar{R}$
$ar=R_\sigma$
r=const.
$ar=H^{-1}$
$\mathrm{d}N$
0
N
$a \times r$
alternating zoom–in
linear perturbations

[Tomberg'24]

Alternating zoom-in scheme

$$\tilde{\phi}(N_+) = \phi(N) + \mu[\phi(N)]\mathrm{d}N\,,$$
$$\phi(N_+) = \tilde{\phi}(N_+) + \sigma[\tilde{\phi}(N_+)]\sqrt{\mathrm{d}N}\,\xi_N$$

Taylor-expand again:

$$\phi(N_+) = \phi(N) + \mu[\phi(N)]\mathrm{d}N + \sigma[\phi(N)]\sqrt{\mathrm{d}N}\,\xi_N \;+\; \mathcal{O}(\mathrm{d}N^{3/2})$$

Alternating zoom-in scheme

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Taylor-expand again:

$$\phi(N_+) = \phi(N) + \mu[\phi(N)]\mathrm{d}N + \sigma[\phi(N)]\sqrt{\mathrm{d}N}\,\xi_N \;+\; \mathcal{O}(\mathrm{d}N^{3/2})$$

Identical to Itô!

Difference Planck-suppressed

$$\mu(\phi) = -\frac{V'(\phi)}{3H^2(\phi)} = -\frac{V'(\phi)M_{\rm Pl}^2}{V(\phi)},$$

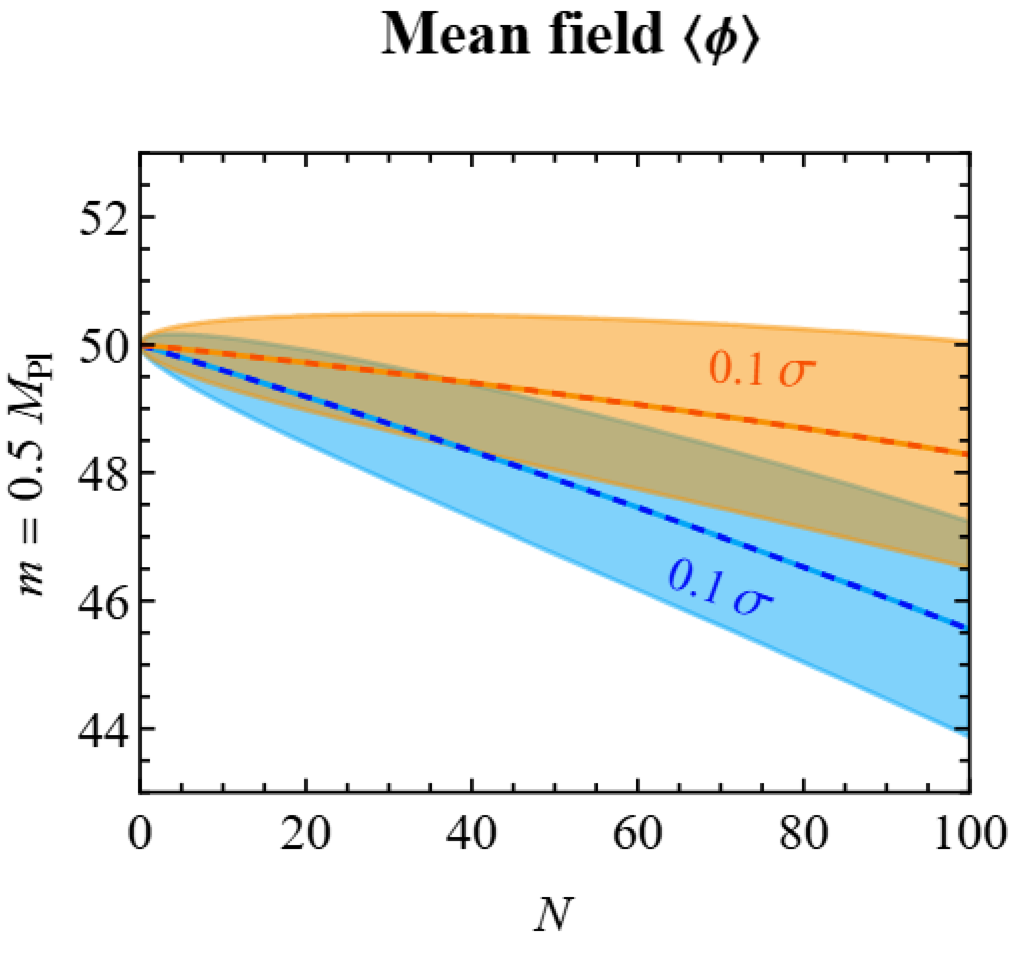
$$\frac{\sigma(\phi)}{2}\frac{\partial\sigma(\phi)}{\partial\phi} = \frac{1}{4}\frac{\partial\sigma^2(\phi)}{\partial\phi} = \frac{1}{16\pi^2}\frac{\partial H^2(\phi)}{\partial\phi} = \frac{V'(\phi)}{48\pi^2 M_{\rm Pl}^2}$$

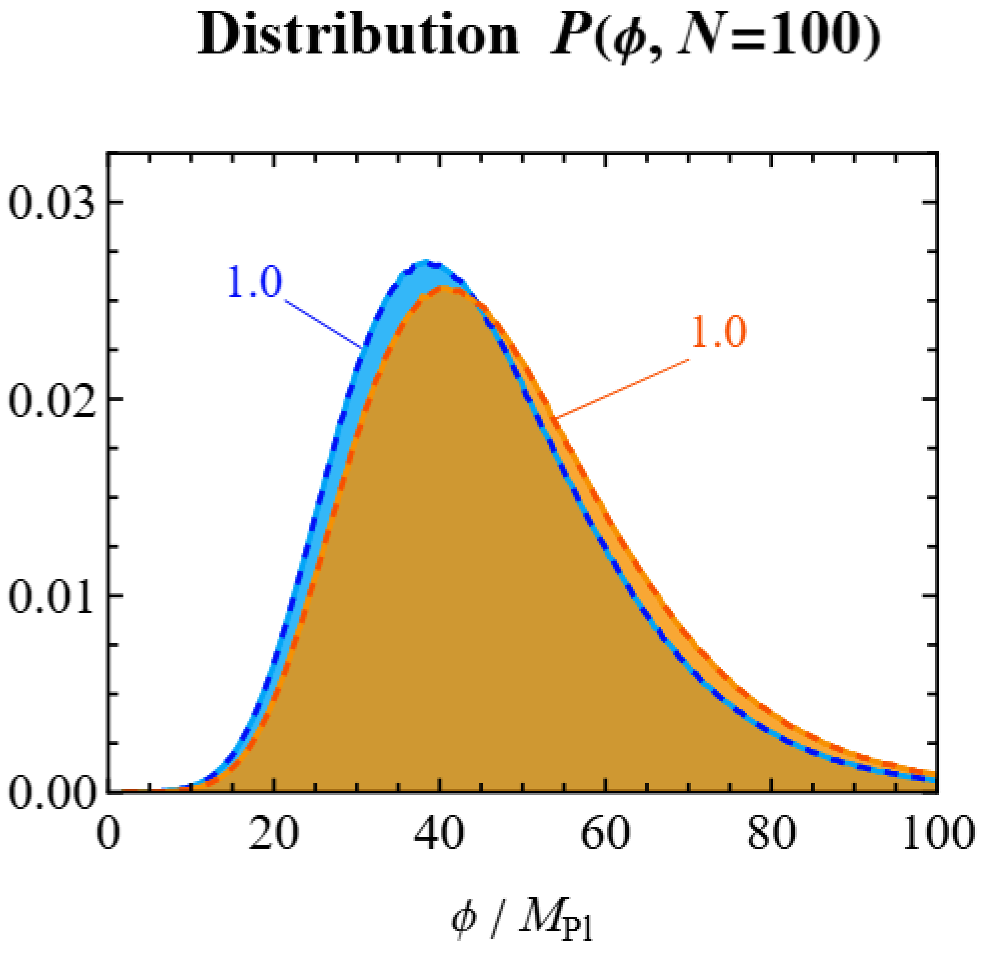
$$\left|\frac{\sigma}{2}\frac{\partial\sigma}{\partial\phi}\right| \ll |\mu| \quad\Longleftrightarrow\quad H \ll 4\pi M_{\rm Pl} \quad\Longleftrightarrow\quad V \ll 48\pi^2 M_{\rm Pl}^4$$

Example evolution

$$V(\phi) = \frac{1}{2}m^2\phi^2 \quad \Longrightarrow \quad \mu(\phi) = -\frac{2M_{\rm Pl}^2}{\phi}\,, \quad \sigma(\phi) = \frac{m\phi}{2\sqrt{6}\pi M_{\rm Pl}}$$

Example evolution





Itô Alternating Stratonovich A Stratonovich B

Self-consistent noise

Solve noise modes in stochastic background:

$$\Phi_i' = \mu_i(\Phi_j, N) + \sigma_i^\alpha(\Phi_j, N)\xi_\alpha(N)\,, \qquad \langle \xi_\alpha(N)\xi_\beta(N')\rangle = \delta_{\alpha\beta}\delta(N-N')$$

$$\Phi_i = \begin{bmatrix} \phi_R \\ \pi_R \\ \delta\phi_k \\ \delta\pi_k \end{bmatrix}, \quad \mu_i = \begin{bmatrix} \pi_R \\ \mu_{\pi_R}(\phi_R, \pi_R) \\ \delta\pi_k \\ \mu_{\delta\pi_k}(\delta\phi_k, \delta\pi_k, \phi_R, \pi_R, N) \end{bmatrix}, \quad \sigma_i^\alpha = \begin{bmatrix} \sigma^\alpha_{\delta\phi_R}(\delta\phi_k, \delta\pi_k, N) \\ \sigma^\alpha_{\delta\pi_R}(\delta\phi_k, \delta\pi_k, N) \\ 0 \\ 0 \end{bmatrix}$$

Self-consistent noise

$$\mu_{\pi_R} = -\left(3 - \frac{1}{2}\pi_R^2\right)\left(\pi_R + \frac{V'(\phi_R)}{V(\phi_R)}\right),$$

$$\mu_{\delta\pi_k} = -\left(3 - \frac{1}{2}\pi_R^2\right)\left(\delta\pi_k + \pi_R^2\delta\phi_k + \frac{1}{V(\phi_R)}\left[\frac{k^2}{a(N)^2} + 2\pi_R V'(\phi_R) + V''(\phi_R)\right]\delta\phi_k\right)$$

$$\sigma_{\delta\phi_R} = \sqrt{\frac{k_\sigma^3}{2\pi^2}}|\delta\phi_{k_\sigma}(N)|, \qquad \sigma_{\delta\pi_R} = \frac{\delta\pi_{k_\sigma}}{\delta\phi_{k_\sigma}}\sigma_{\delta\phi_R}$$

Self-consistent noise

Problematic extra drift terms are all zero:

$$\begin{aligned}\sigma_i^\alpha[\Phi_j(N_+), N_+] &\sim \frac{\partial}{\partial \Phi_l}\sigma_i^\alpha[\Phi_j(N), N] \times \mathrm{d}\Phi_l \\ &\sim \frac{\partial}{\partial \Phi_l}\sigma_i^\alpha[\Phi_j(N), N] \times \sigma_l^\alpha[\Phi_j(N), N]\sqrt{\mathrm{d}N}\xi_{\alpha,N} \\ &\sim 0\end{aligned}$$

Self-consistent noise

Problematic extra drift terms are all zero:

$$\sigma_i^\alpha[\Phi_j(N_+), N_+] \sim \frac{\partial}{\partial \Phi_l}\sigma_i^\alpha[\Phi_j(N), N] \times \mathrm{d}\Phi_l$$

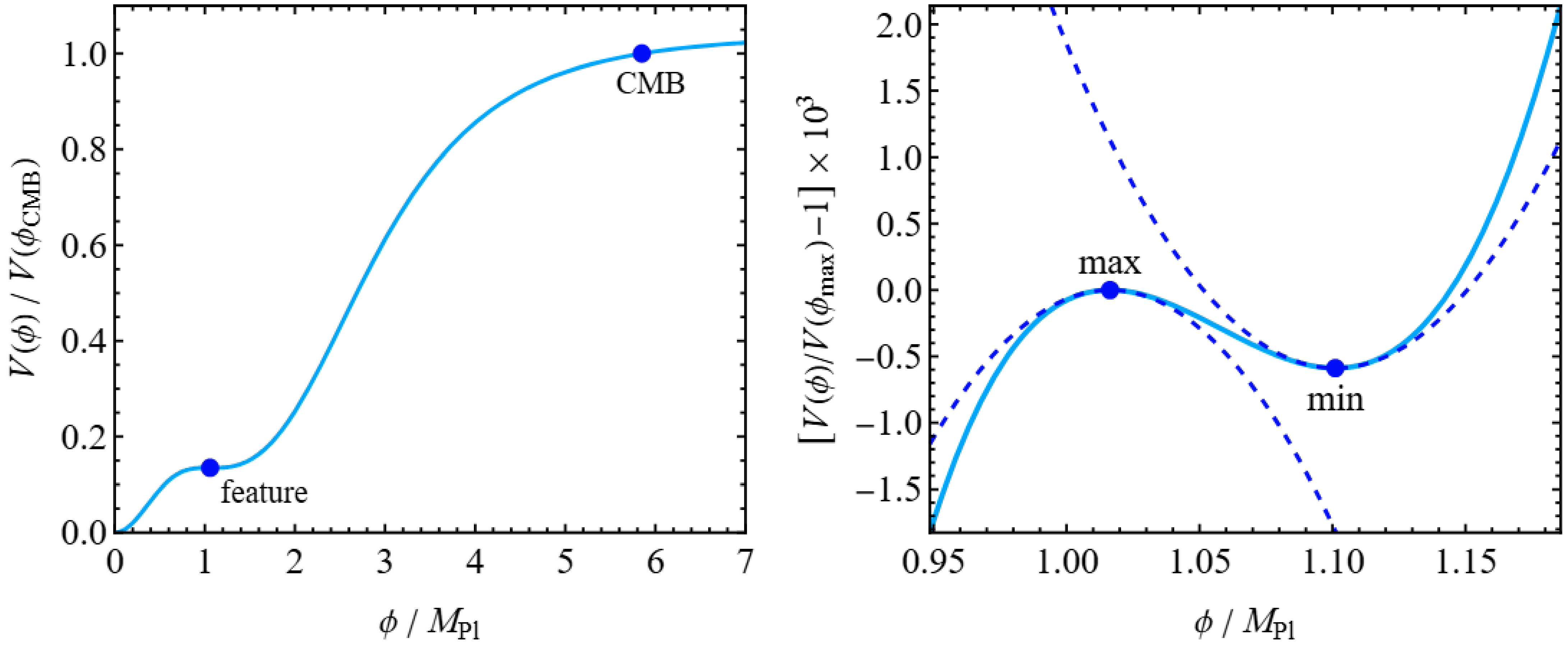
$$\sim \frac{\partial}{\partial \Phi_l}\sigma_i^\alpha[\Phi_j(N), N] \times \sigma_l^\alpha[\Phi_j(N), N]\sqrt{\mathrm{d}N}\xi_{\alpha,N}$$

$$\sim 0$$

Itô and Stratonovich become equal!

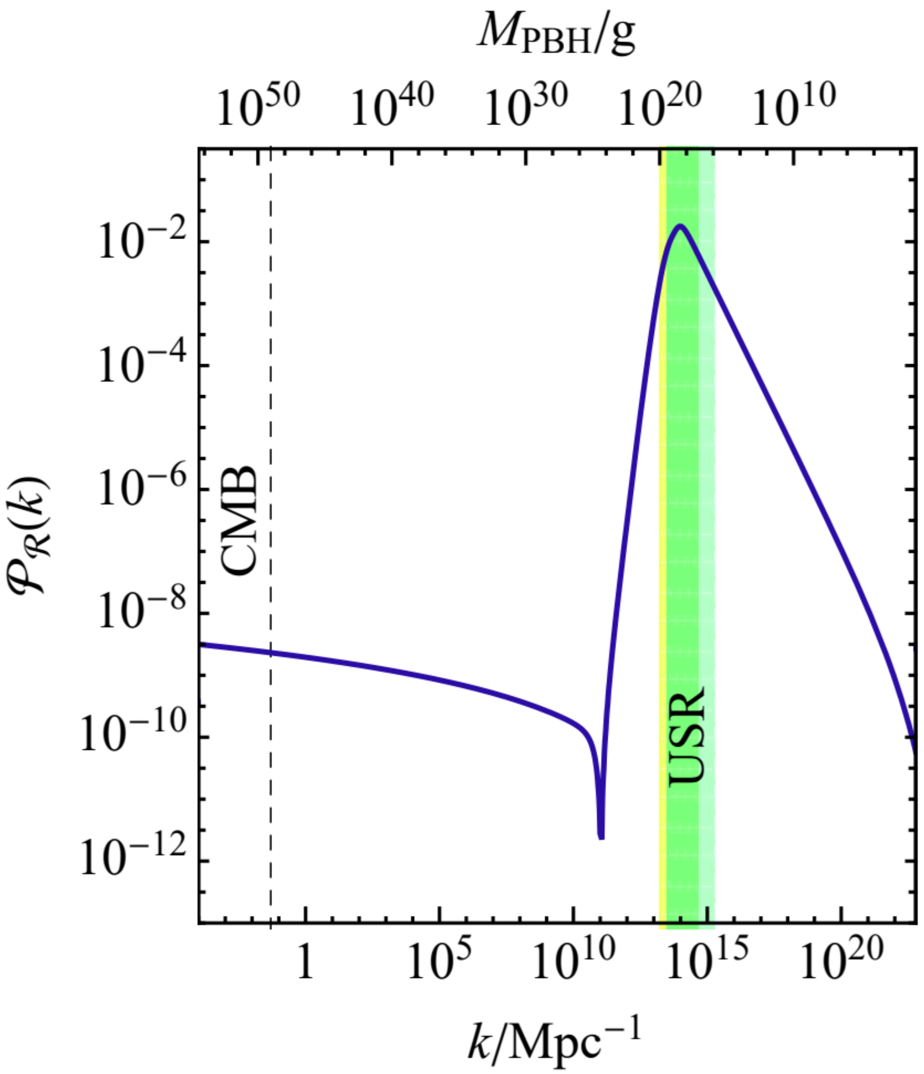
Inflection point inflation

[Tomberg & Dimopoulos '25], under preparation



Eternal inflation

[Karam et al. '22]



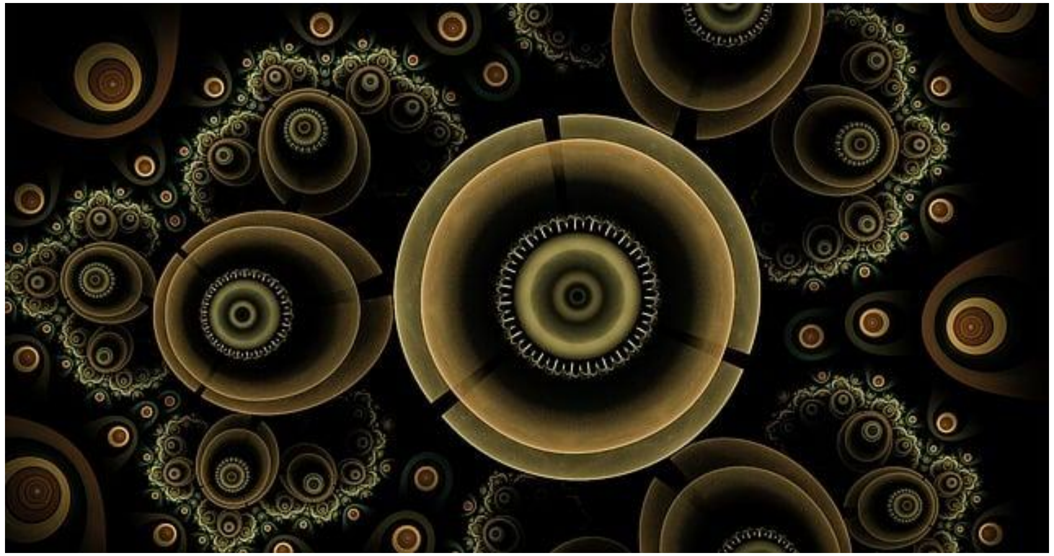
Large quantum fluctuations counteract classical drift

Inflating regions grow fast

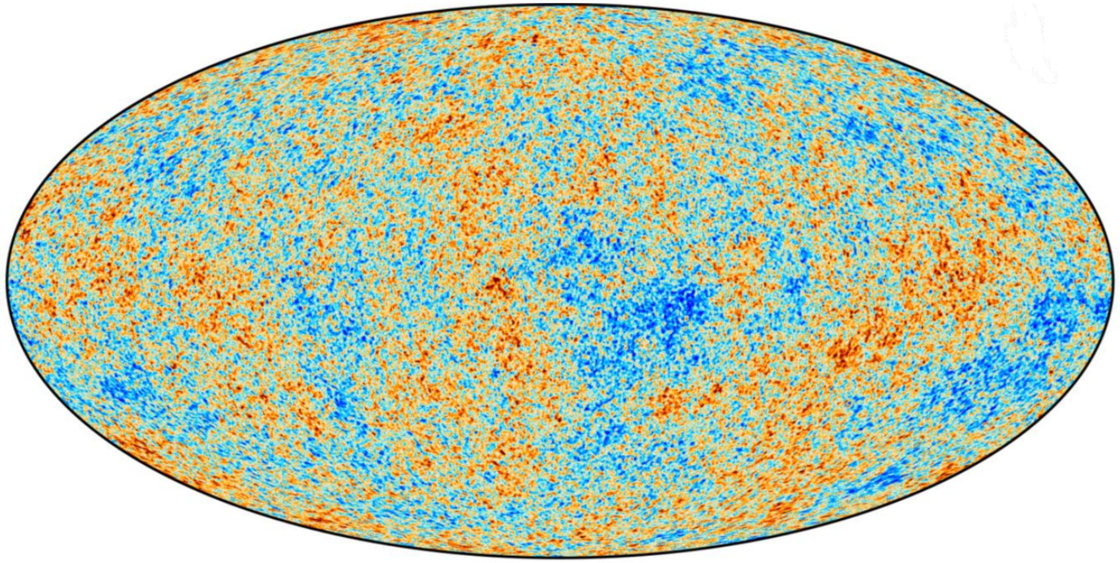
Inflation is eternal: 'most' of the Universe always inflating

Why interesting?

Global structure?



Observational predictions?



Eternal inflation?

Heuristic estimate:

quantum fluctuations dominate over classical drift

$$\frac{\sigma}{\mu} \sim \frac{H}{\sqrt{\epsilon_V}} \sim \mathcal{P}_{\mathcal{R}} \gtrsim 1$$

We have: $\mathcal{P}_{\mathcal{R}} < 1$ so no eternal inflation?

However, there are (attractor) trajectories with $\mathcal{P}_{\mathcal{R}} > 1$ near the extrema

Eternal inflation?

Precise definition:

Inflating volume non-zero at late times

$$\lim_{N\to\infty} \langle V\rangle_N \equiv \lim_{N\to\infty} \int_{\phi_{\rm end}}^{\infty} e^{3N} P(\phi, N)\mathrm{d}\phi > 0$$

Solving P

Fokker-Planck equation:

$$\partial_N P(\phi, N) = \underbrace{\partial_\phi \left[\partial_\phi \left(\frac{1}{2}\sigma^2(\phi) P(\phi, N)\right) - \mu(\phi) P(\phi, N)\right]}_{\equiv \mathcal{L}_\phi P(\phi, N)}$$

with absorbing boundary condition $P(\phi_{\rm end}, N) = 0$ at end-of-inflation hypersurface

Solving P

Eigenfunction decomposition:

$$\mathcal{L}_\phi u_n(\phi) = -\lambda_n u_n(\phi)$$

$$P(\phi, N) = \sum_n a_n(N) u_n(\phi)$$

$$a_n(N) = a_n(0) e^{-\lambda_n N}$$

See also [Ezquiaga et al. '18] etc.

Solving P

Eigenfunction decomposition:

$$\implies \lim_{N\to\infty} P(\phi, N) \sim e^{-\lambda_1 N}$$

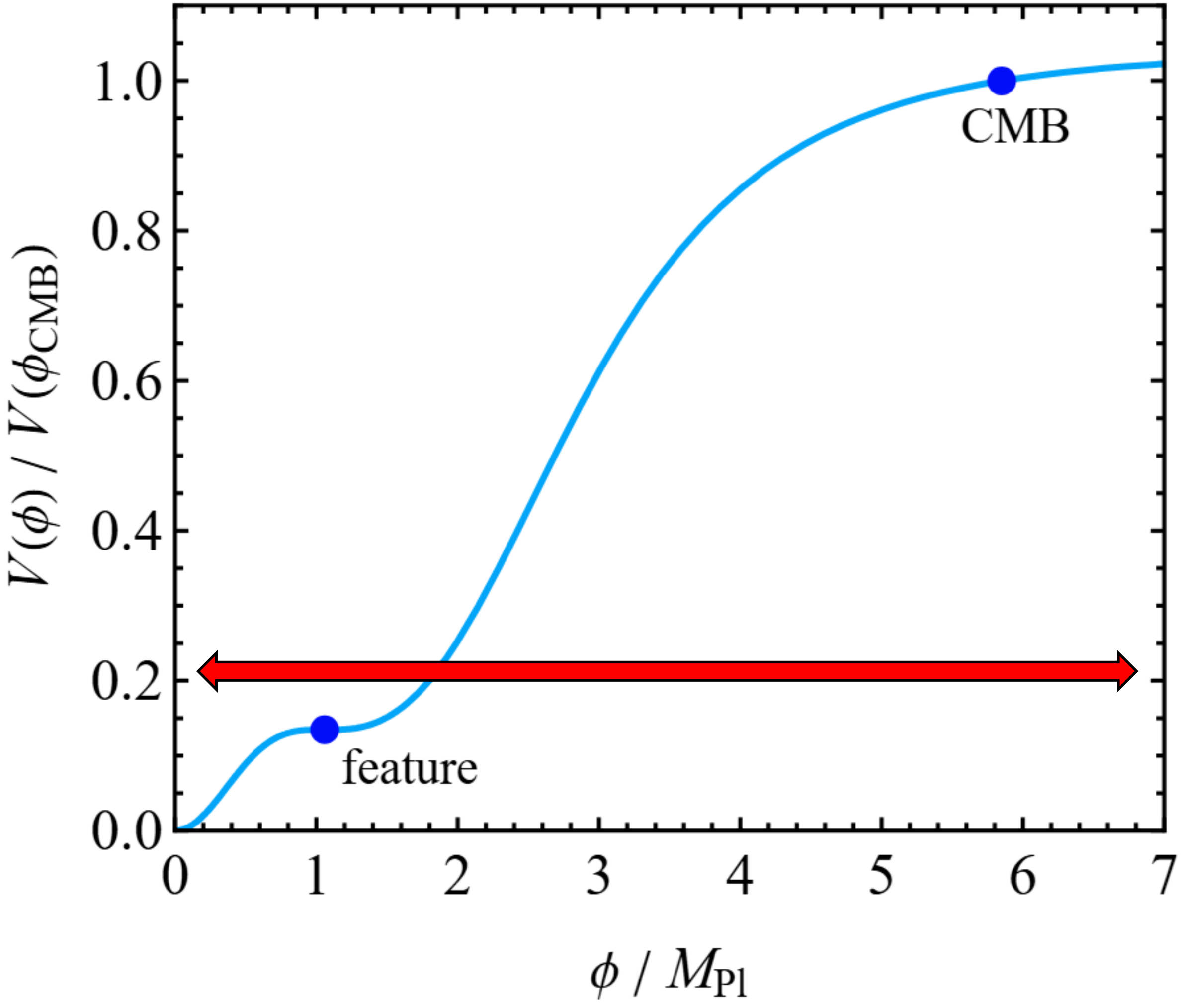
$$\implies \lim_{N\to\infty} \langle V \rangle_N \sim e^{(3-\lambda_1)N}$$

Solving P

$$\text{eternal inflation} \quad \Longleftrightarrow \quad \lambda_1 \leq 3$$

Practical problems

Features in potential
on a short field interval



Practical problems

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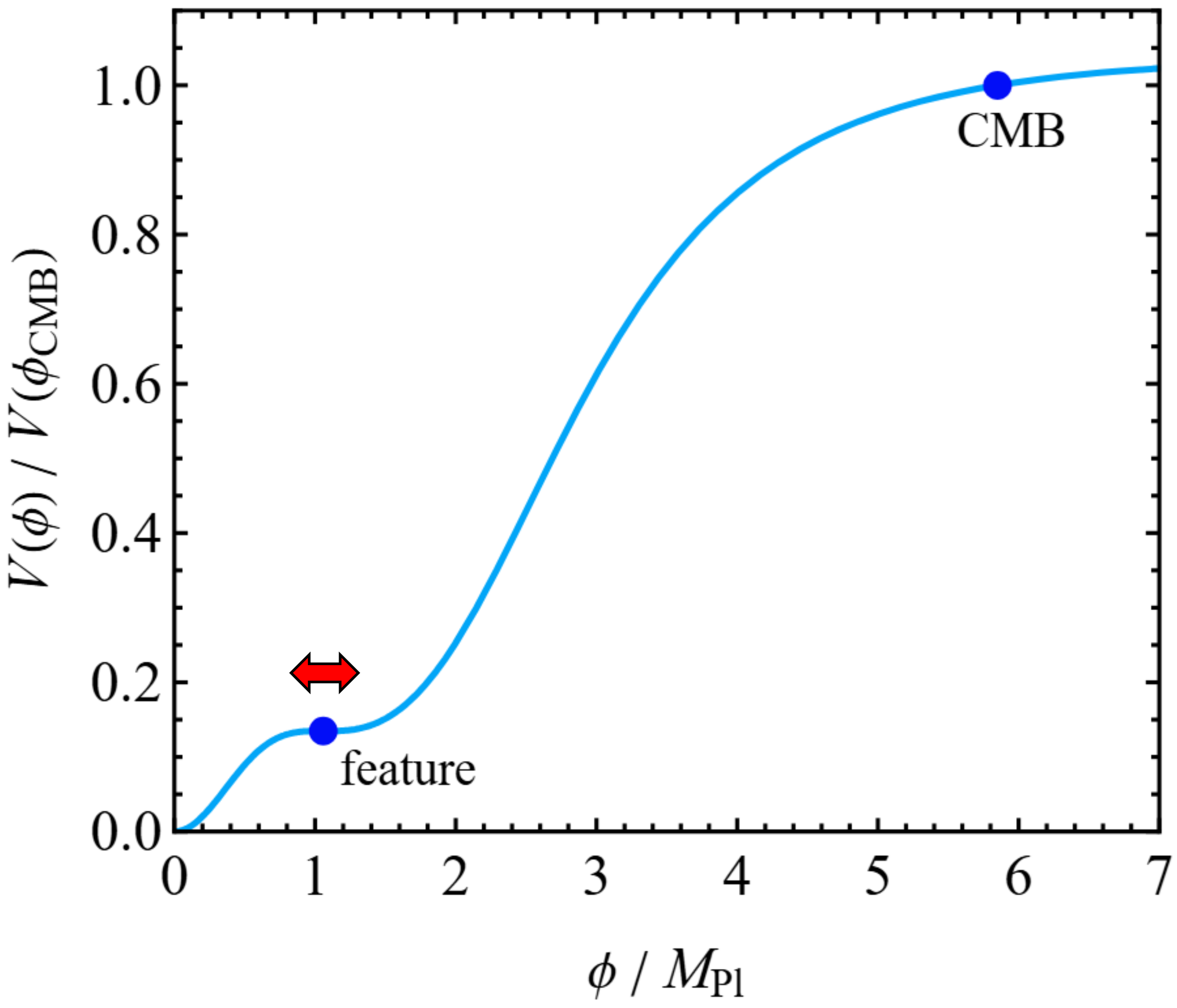
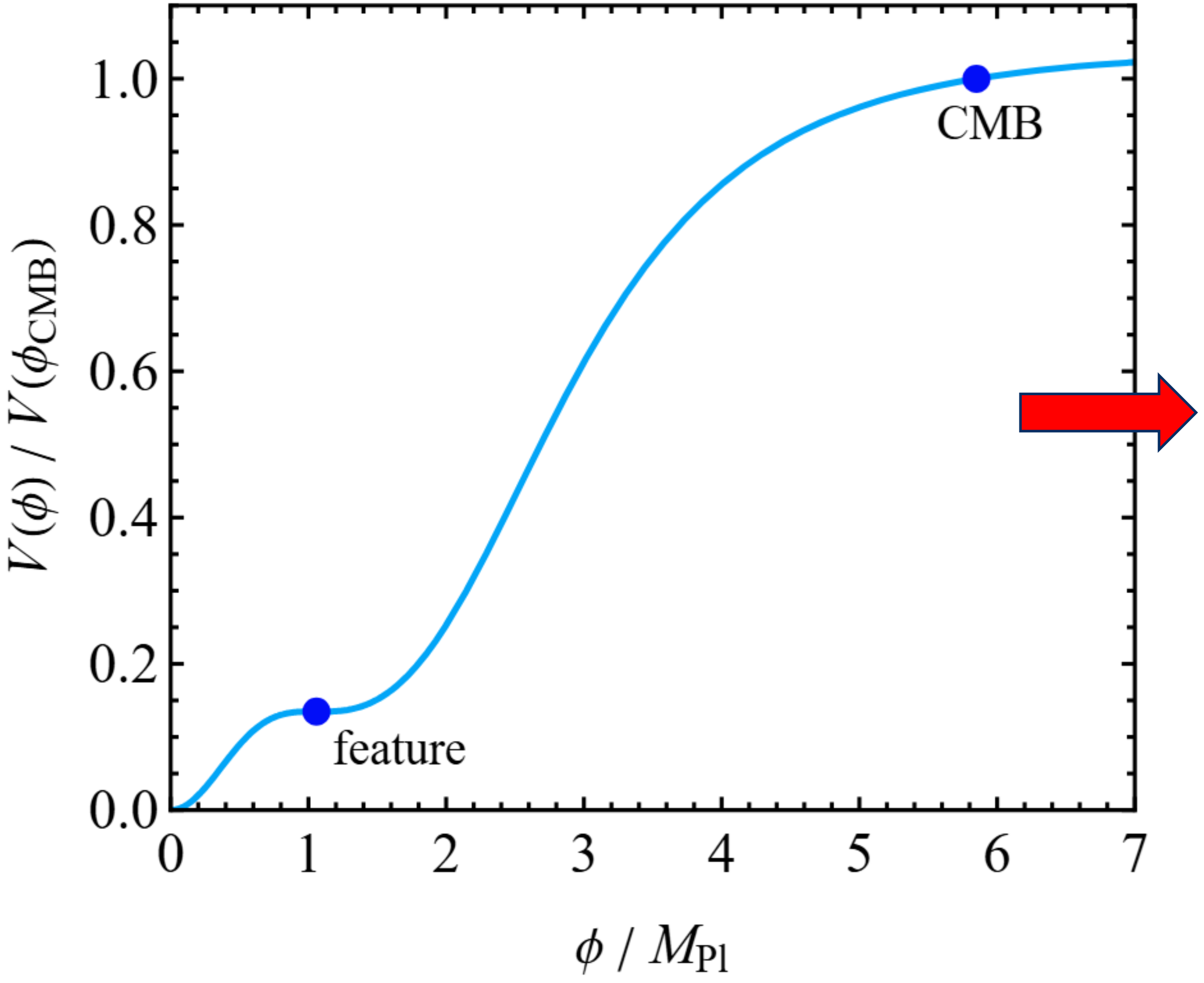


Practical problems

Features in potential
on a short field interval

Usually eternal inflation
at large field values

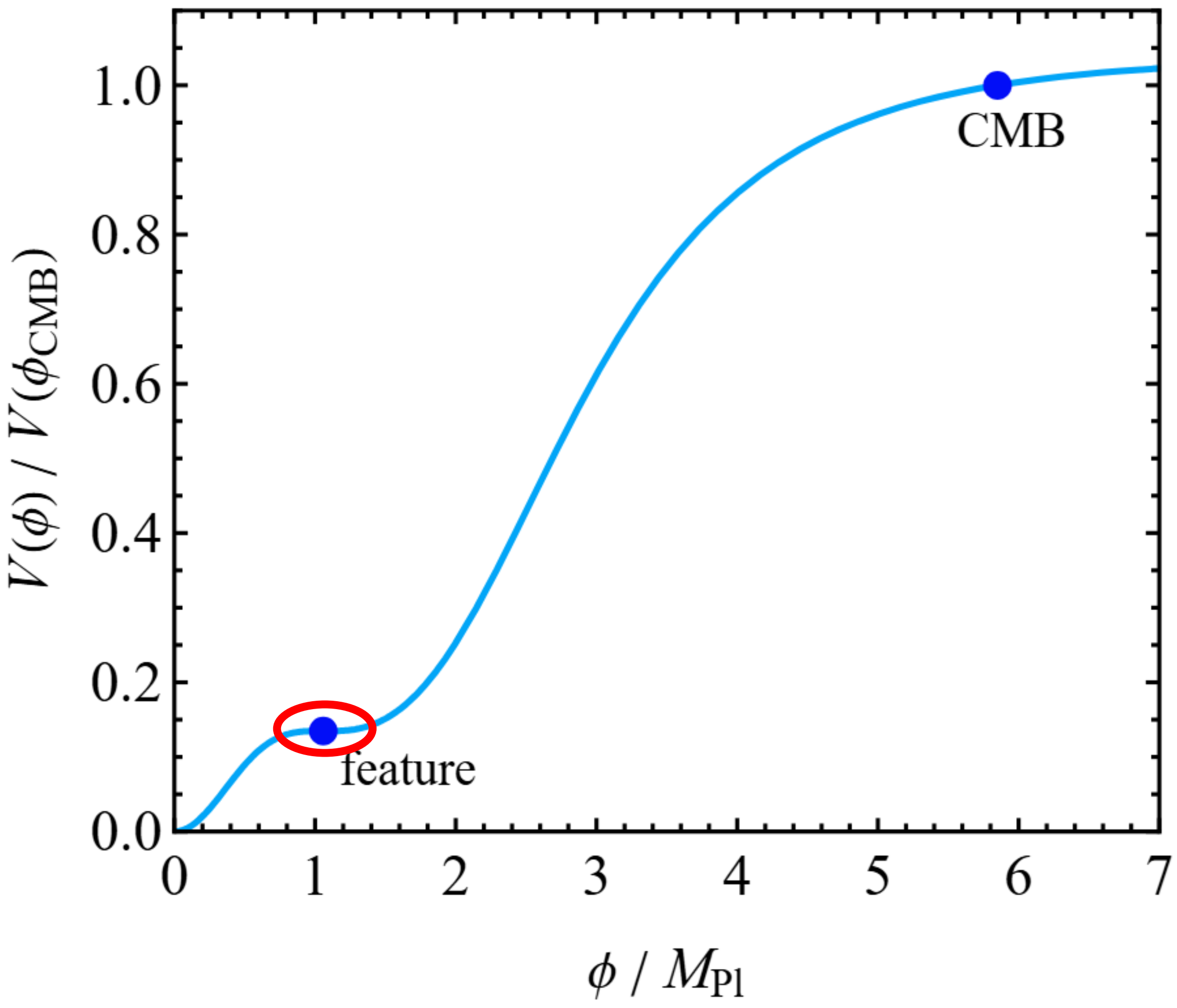


Practical problems

Features in potential on a short field interval

Usually eternal inflation at large field values

Focus on the feature



Take a subet of paths

Small field interval: $\phi \in [\phi_a, \phi_b]$

Look at paths that spend 'most of the time' there (can be made rigorous)

Solve $\lambda_{1,\text{local}}$ with absorbing boundaries at ϕ_a, ϕ_b

Take a subet of paths

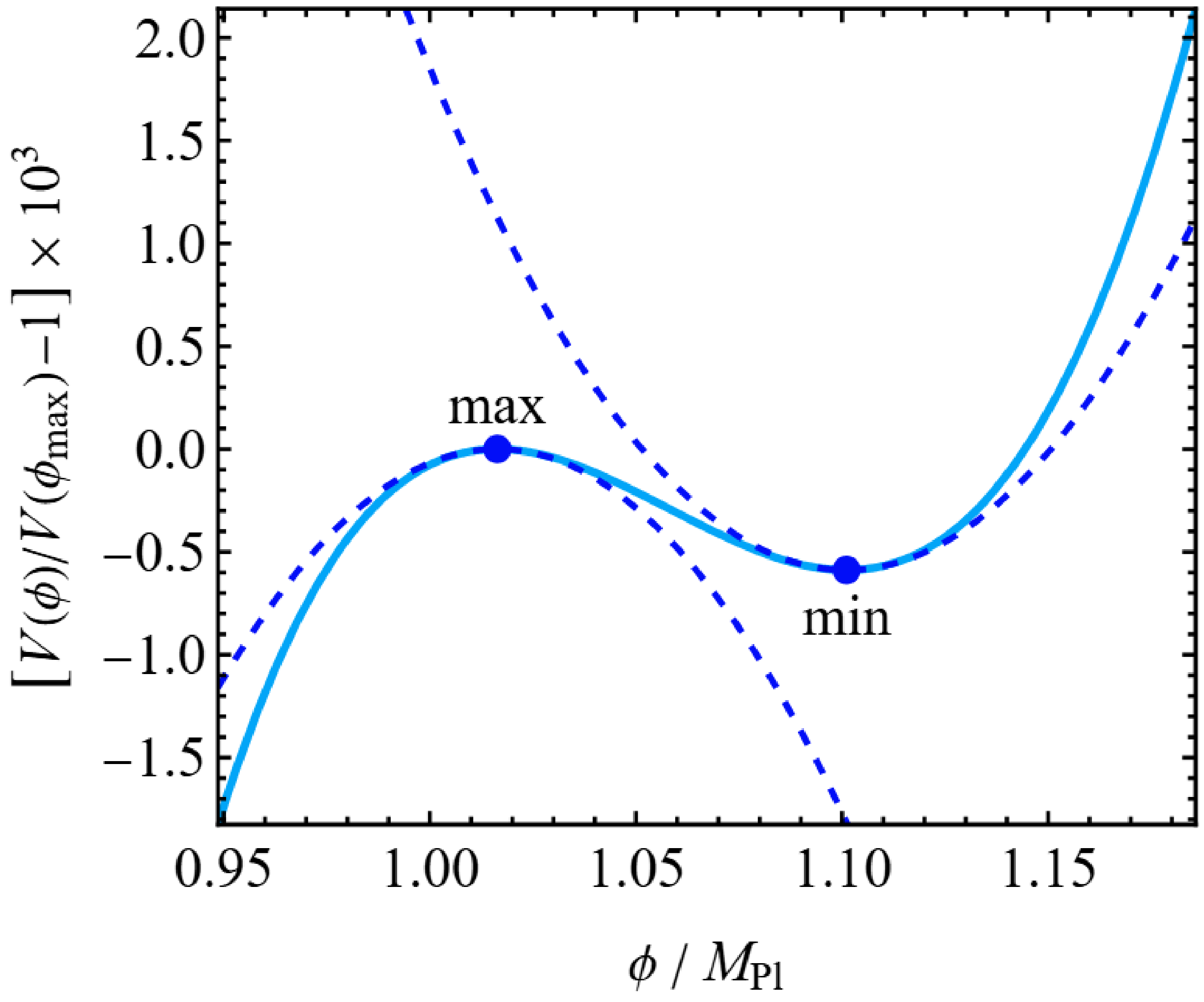
Condition for eternally inflation from *these* paths: $\lambda_{1,\mathrm{local}} \leq 3$

Can show: $\lambda_{1,\mathrm{global}} \leq \lambda_{1,\mathrm{local}}$

Parabolic approximation

Near extrema:

$$V(\phi) = V_0\left(1 + \frac{1}{2}\eta_V(\phi - \phi_i)^2\right)$$



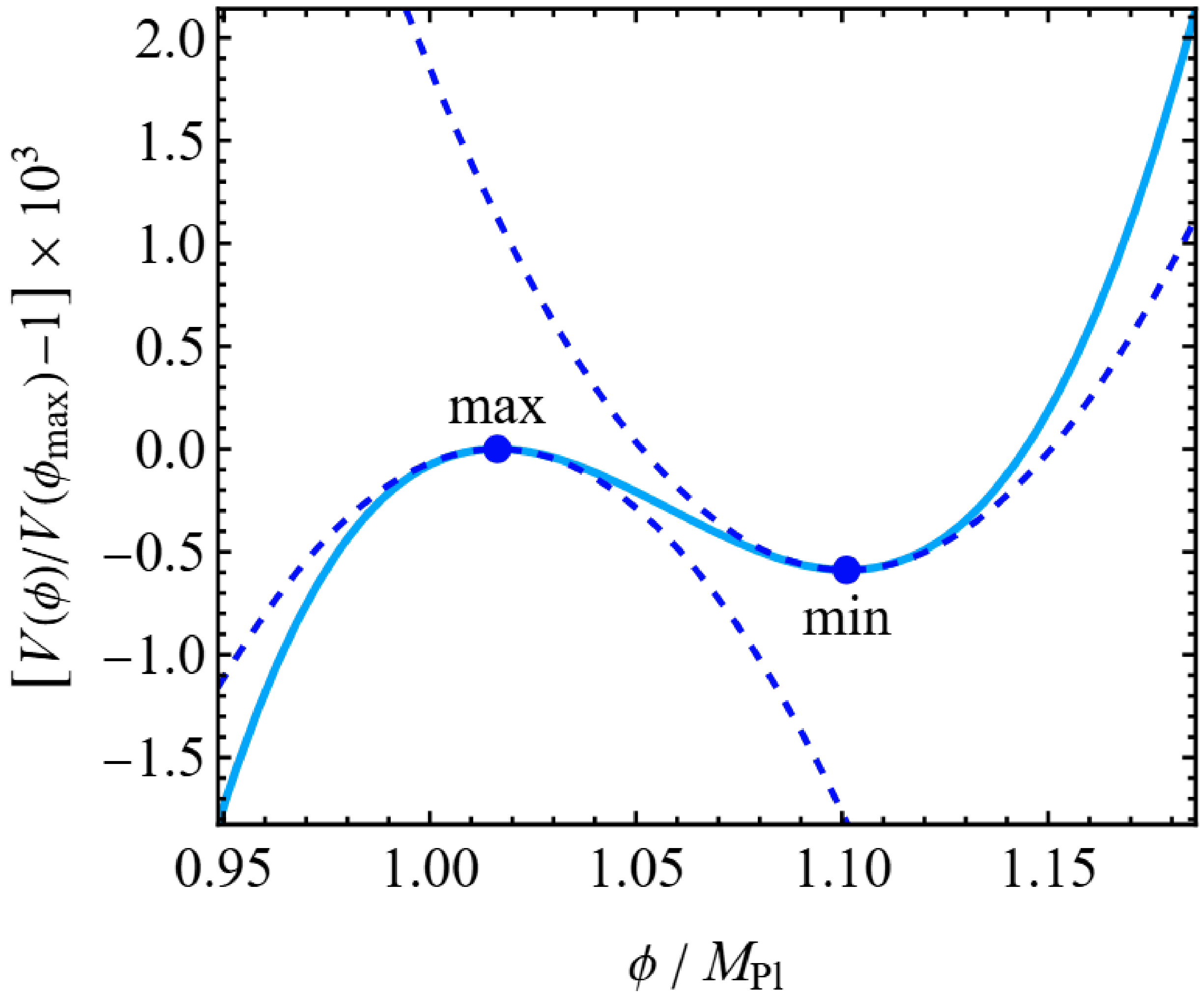
Parabolic approximation

Constant-roll inflation:

Tomberg 2023

$$H = \sqrt{\frac{V_0}{3}} = \text{const.}$$

$$\eta_H = \frac{3}{2}\left(1 - \sqrt{1 - \frac{4}{3}\eta_V}\right)$$
$$= \text{const.}$$



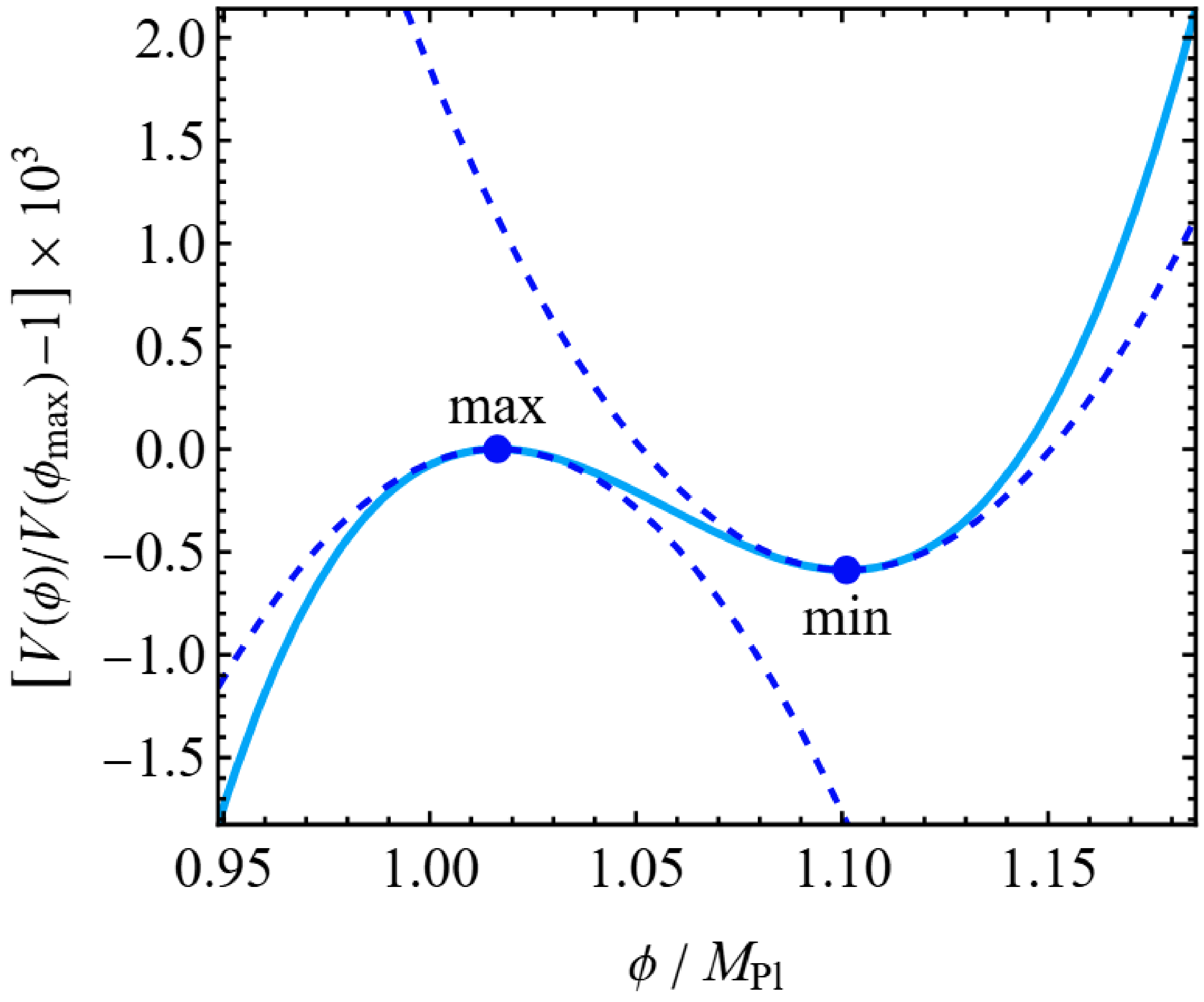
Parabolic approximation

Constant-roll inflation:

Tomberg 2023

$$\sigma = \frac{H}{2\pi} \times \frac{\Gamma\left(\frac{3}{2} - \eta_H\right)}{\Gamma\left(\frac{3}{2}\right)} \times \left(\frac{2}{\sigma_c}\right)^{-\eta_H}$$

$$\mu = -\eta_H(\phi - \phi_i)$$



Parabolic approximation

Constant-roll inflation:

Tomberg 2023

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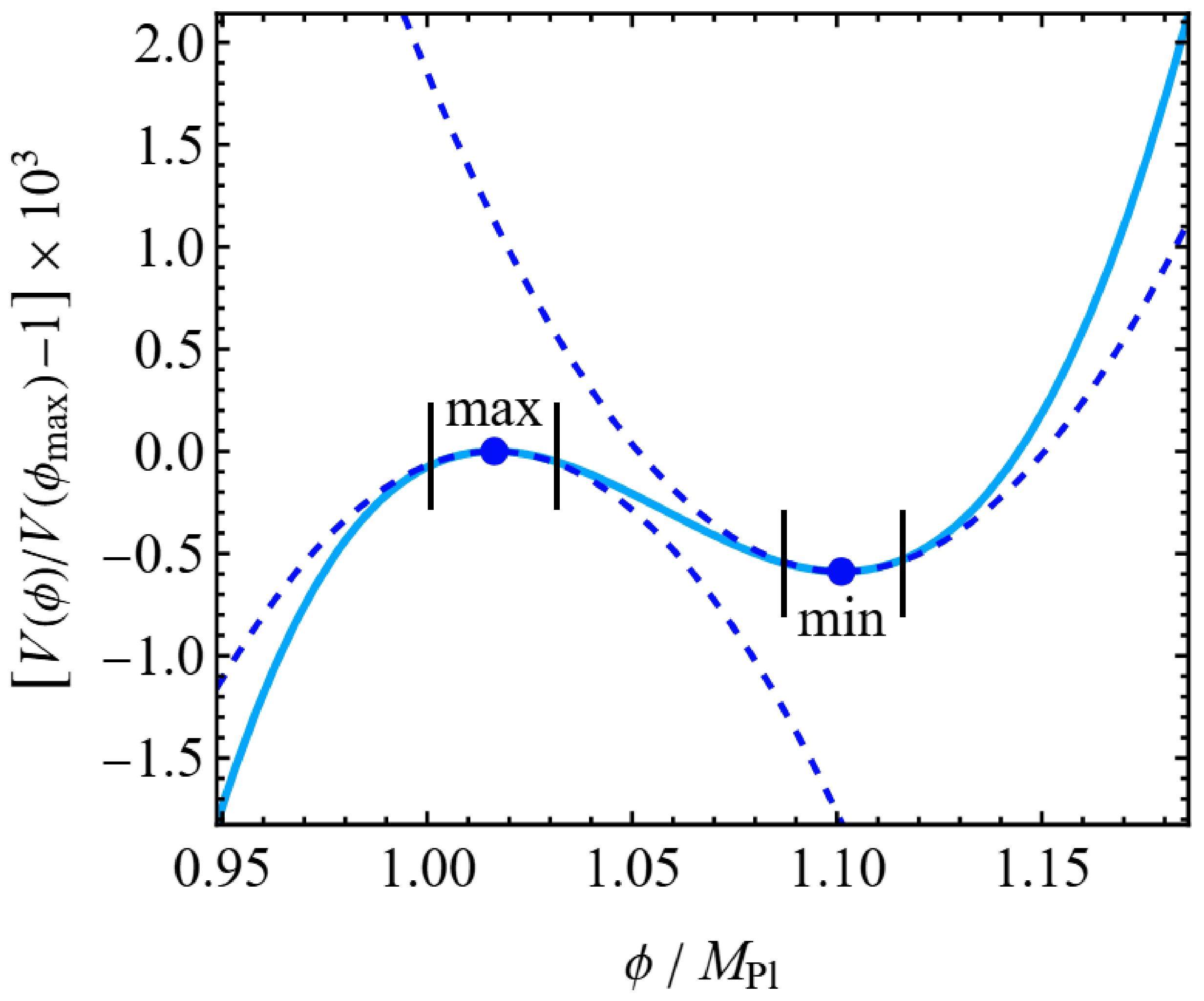
$$\mu = -\eta_H(\phi - \phi_i)$$

Boundaries: $|\phi - \phi_i| < \phi_b$



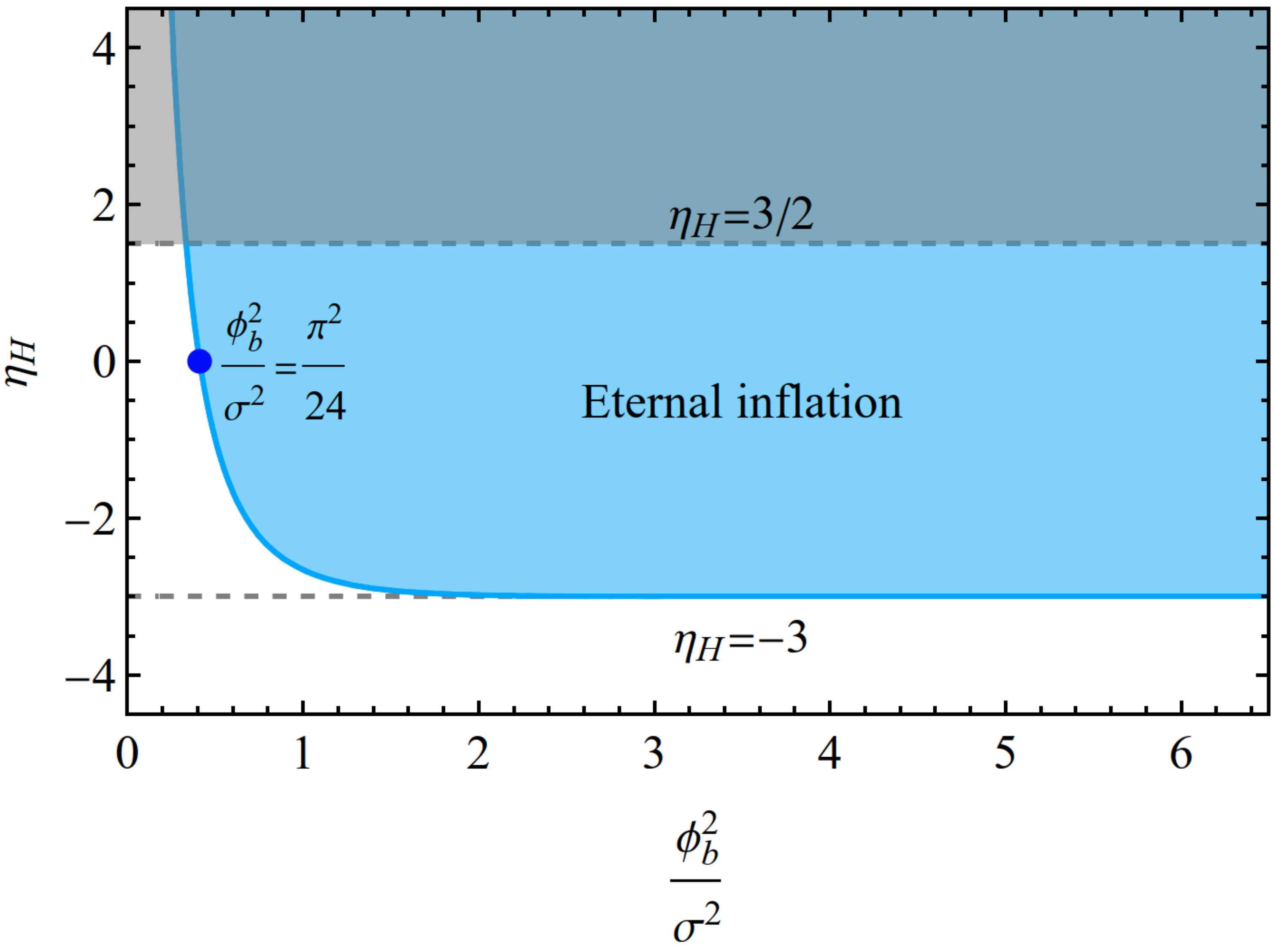
Parabolic approximation

Kummer's equation; lowest eigenvalue from

$$ {}_1F_1\left(-\frac{\lambda_1}{2\eta_H};\frac{1}{2};\frac{\phi_b^2}{\sigma^2}\eta_H\right)=0 $$

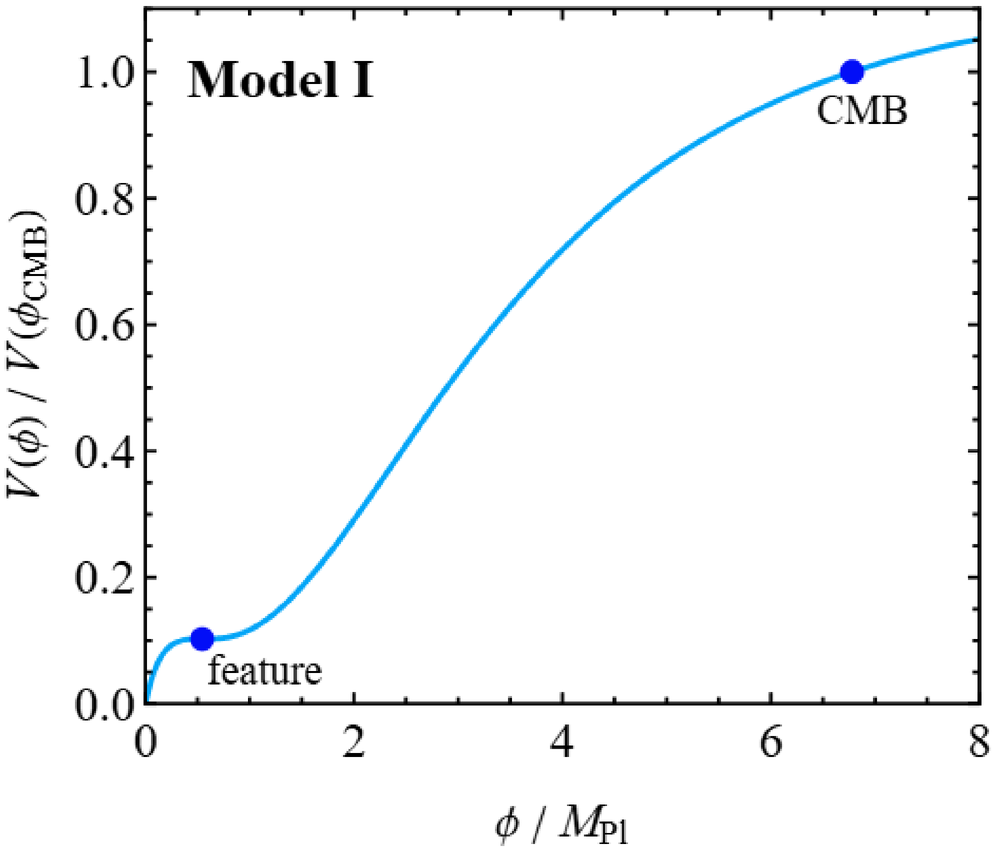
Wide limit: $\phi_b^2 \gtrsim \sigma^2 \implies \lambda_1 \approx \begin{cases} |\eta_H|\,, & \eta_H < 0 \\ 0\,, & \eta_H > 0 \end{cases}$

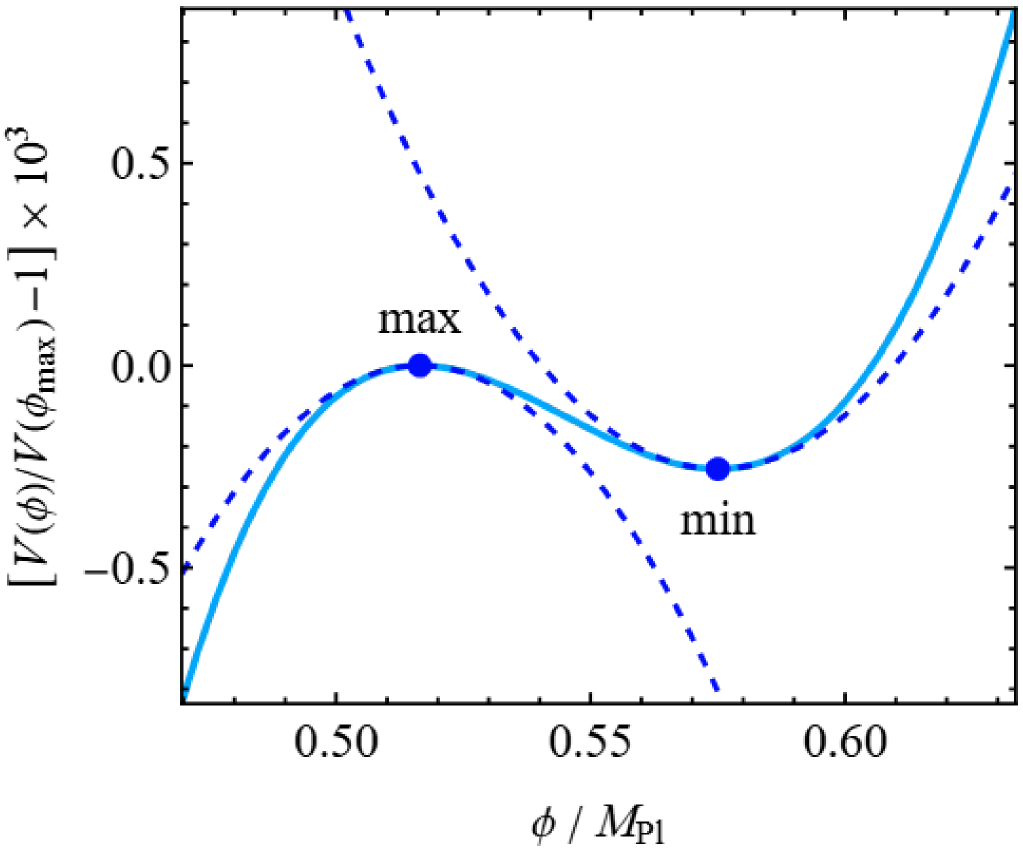
Parabolic approximation



Typical potentials:

[Kannike et al. ’17]



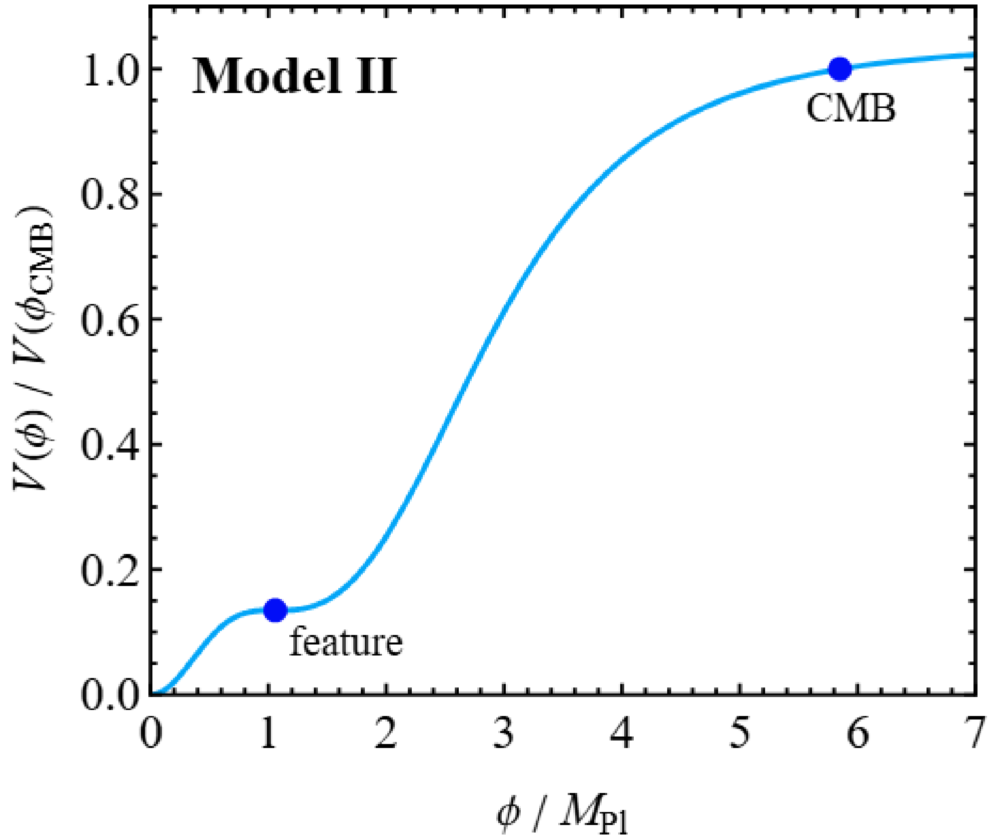


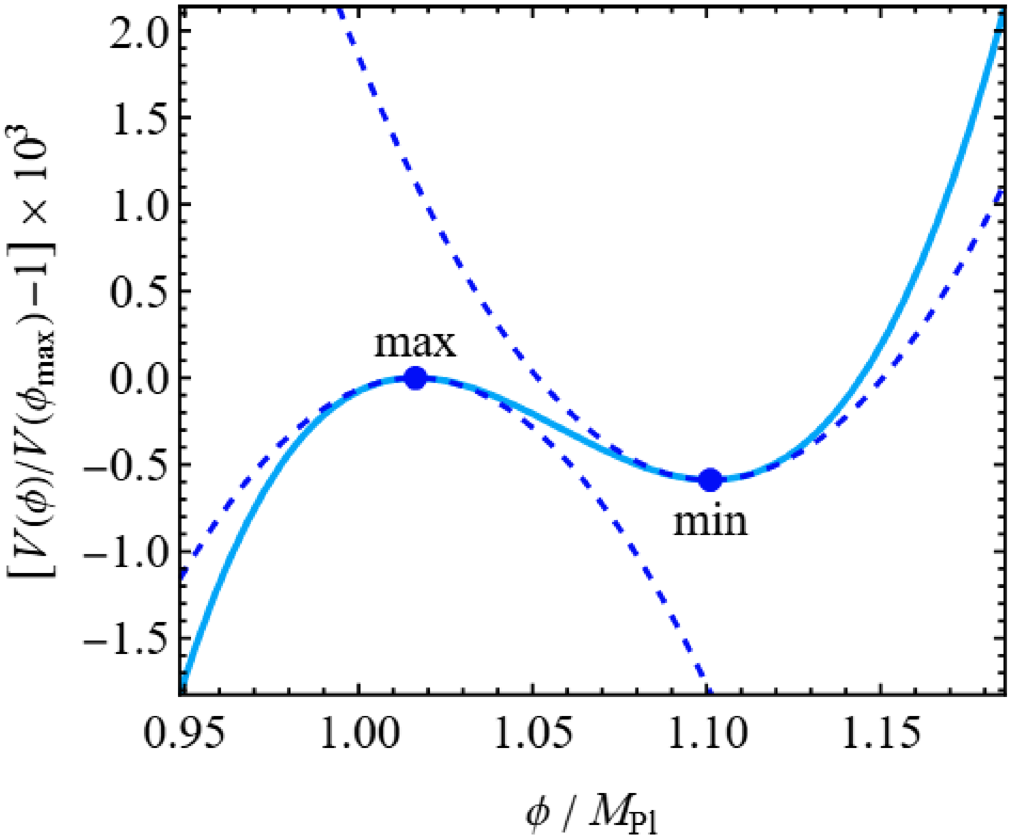
max: $\phi_b^2/\sigma^2 \sim 10^4\,,\ \lambda = -\eta_H = 0.413$

min: $\phi_b^2/\sigma^2 \sim 10^8\,,\ \lambda \approx 0$

Typical potentials:

[Dalianis et al. '18]

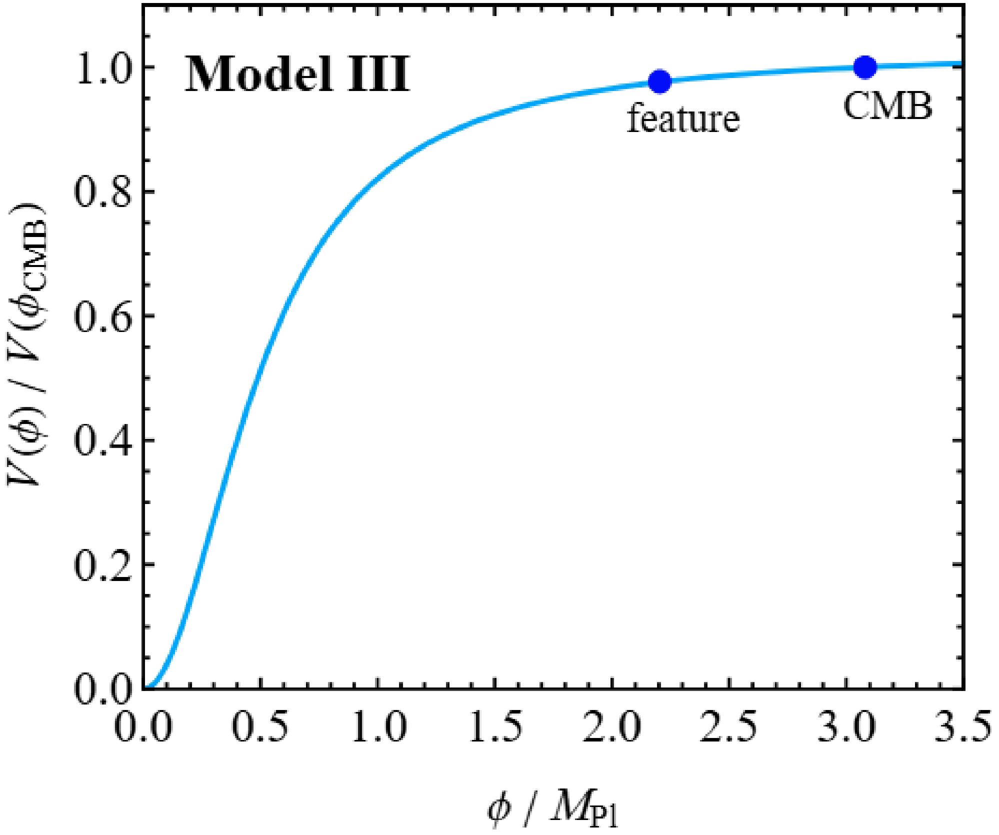


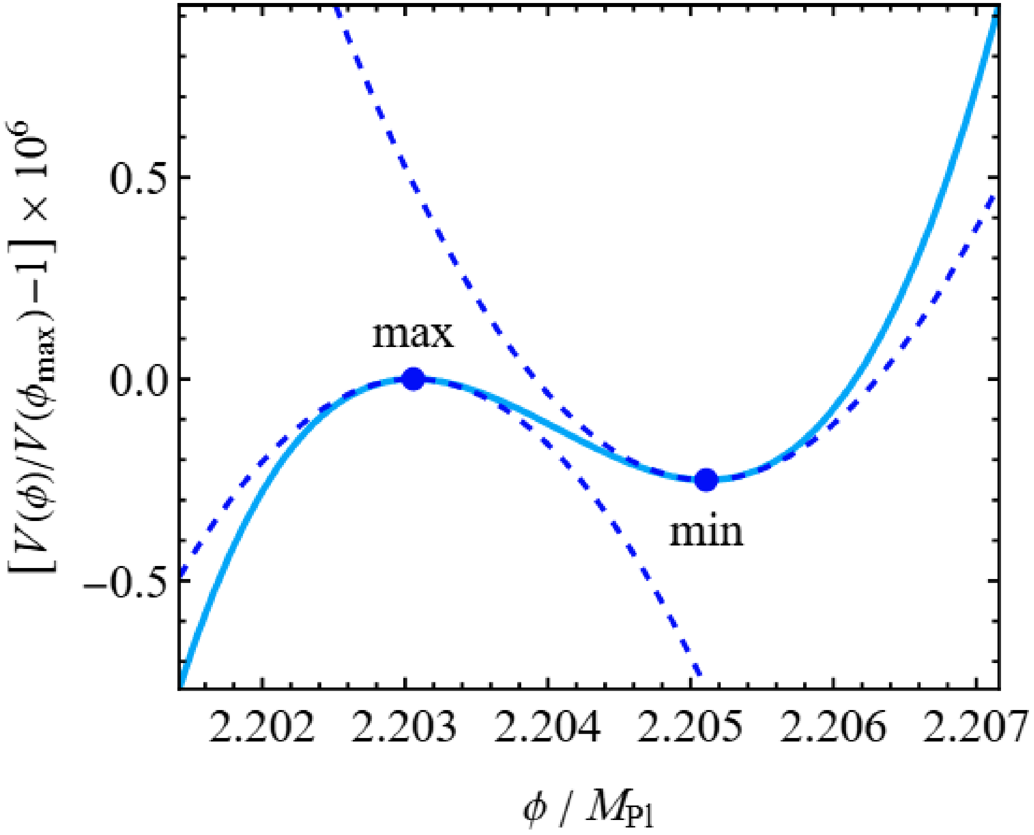


max: $\phi_b^2/\sigma^2 \sim 10^4\,,\ \lambda = -\eta_H = 0.441$ min: $\phi_b^2/\sigma^2 \sim 10^9\,,\ \lambda \approx 0$

Typical potentials:

[Mishra & Sahni '19]





max: $\phi_b^2/\sigma^2 \approx 40\,,\ \lambda = -\eta_H = 0.329$

min: $\phi_b^2/\sigma^2 \sim 10^5\,,\ \lambda \approx 0$

Typical potentials:

Eternal inflation both at the maxima and at the minima

Eternal inflation is generic!

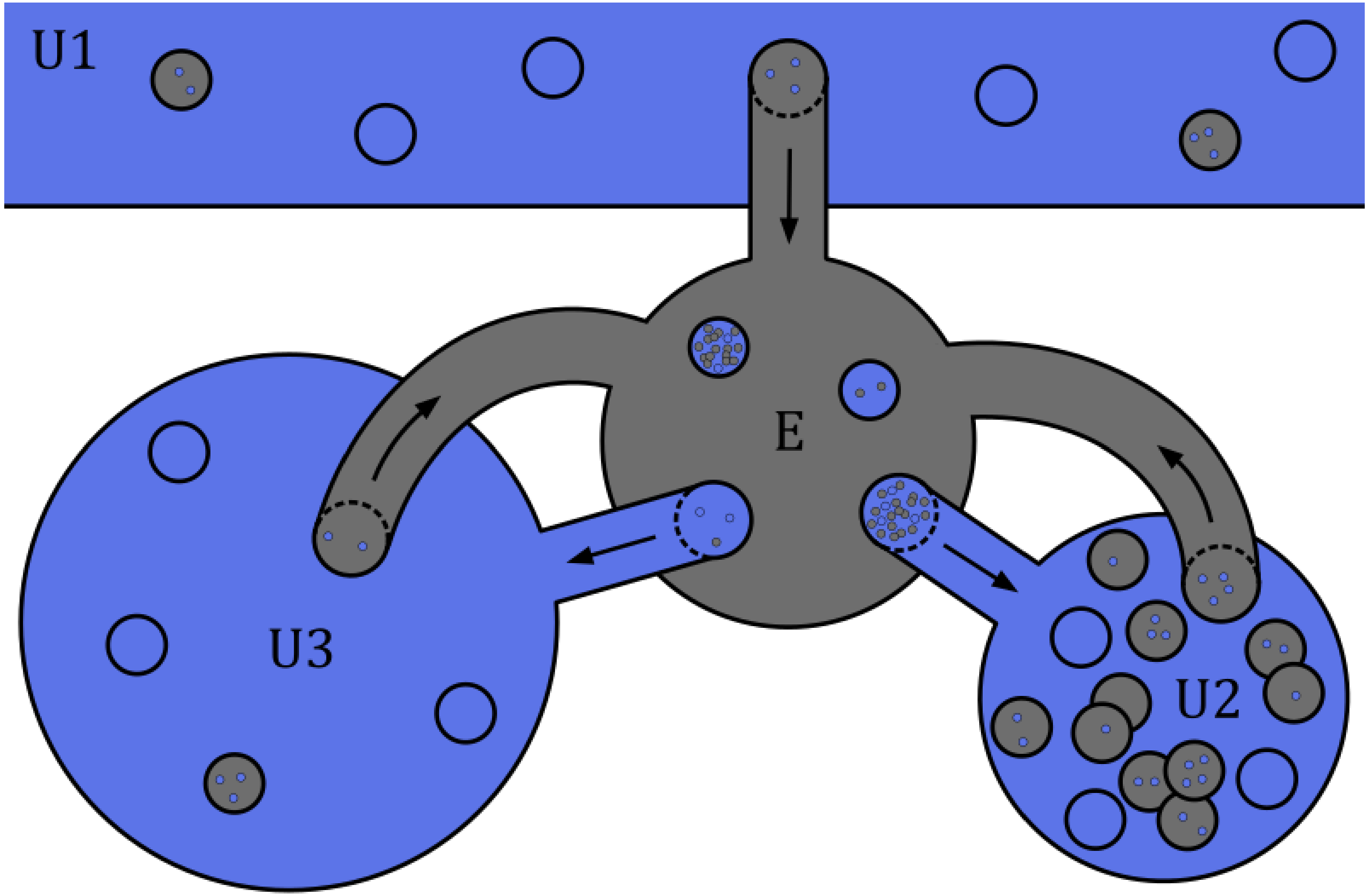
Structure of space-time

Eternal inflation: inside black holes (larger from the inside: type II)

Eternally inflating regions: exit to either side of feature, roll to end-of-inflation surface

These exits dominate!

Structure of space-time



CMB predictions

Exiting towards the end of inflation:
space-time extremely ‘bumpy’ at PBH scale, can’t produce life

Exiting towards the CMB scales:
most likely to not reach all the way there;
space-time ‘bumpy’ at large scales

Questions

Spoiled CMB predictions?

Measure problem?

Initial conditions for inflation?