Stochastic inflation: numerics and constraints

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Concepts

Cosmic inflation

- Accelerating expansion of space in the early universe

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Cosmological perturbations

- Cosmic microwave background, ...

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Primordial black holes (PBHs)

- Dark matter candidate

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Stochastic inflation

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- Includes non-linear effects

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Stochastic inflation

- Includes non-linear effects
- Crucial for the strongest, rarest perturbations

Inflation driven by a scalar field

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - V(\varphi)\right]$$

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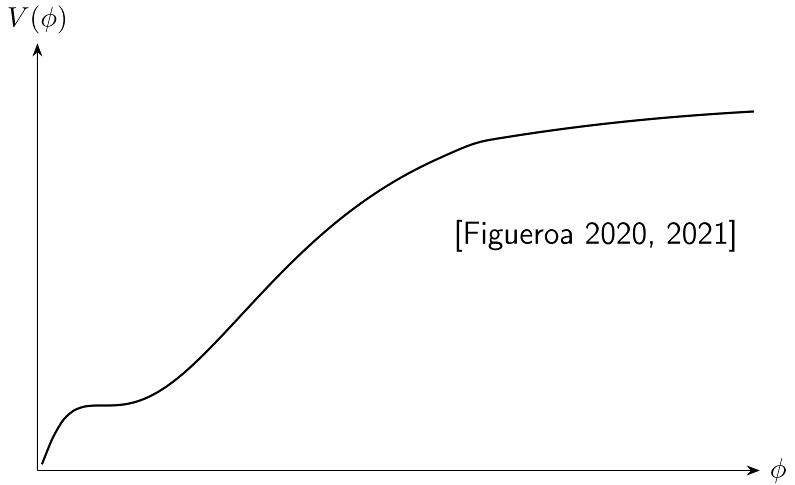
Divide into short-wavelength and coarse-grained parts:

$$\varphi(N,\vec{x}) \equiv \phi(N,\vec{x}) + \delta\phi(N,\vec{x})$$

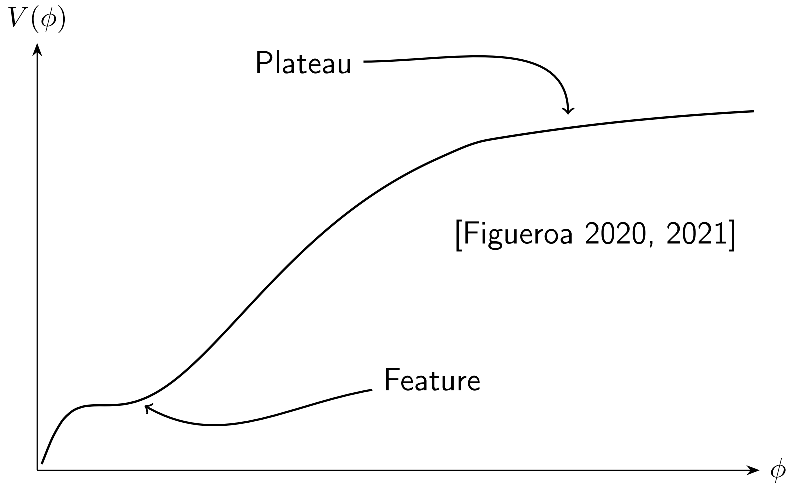
$$= \int_{k<k_\sigma} \frac{\mathrm{d}^3k}{(2\pi)^{2/3}}\phi_k(N)e^{-i\vec{k}\cdot\vec{x}} + \int_{k>k_\sigma} \frac{\mathrm{d}^3k}{(2\pi)^{2/3}}\delta\phi_k(N)e^{-i\vec{k}\cdot\vec{x}}$$

$$k_\sigma \equiv \sigma a H$$

Inflation driven by a scalar field



Inflation driven by a scalar field



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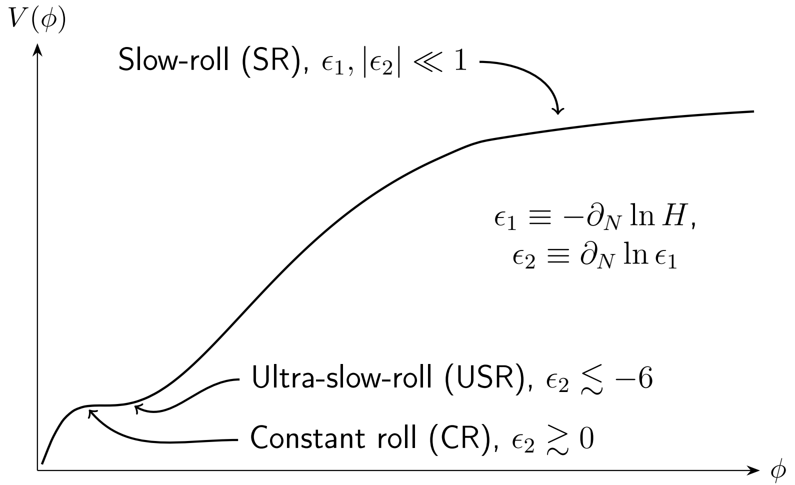

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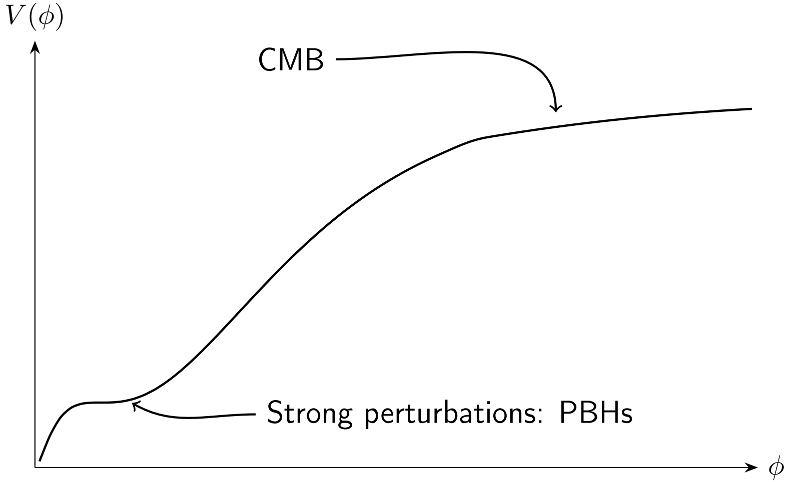

Local background evolves stochastically

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0\,, \qquad 3H^2 = \frac{1}{2}\dot{\phi}^2 + V$$

Local background evolves stochastically

$$\phi'' + \left(3 - \frac{1}{2}\phi'^2\right)\phi' + \frac{V'}{H^2} = 0\,, \qquad \left(3 - \frac{1}{2}\phi'^2\right)H^2 = V$$

Local background evolves stochastically

$$\phi' = \pi\,, \qquad \pi' = -\left(3 - \frac{1}{2}\pi^2\right)\pi - \frac{V'}{H^2}$$

$$\left(3 - \frac{1}{2}\pi^2\right)H^2 = V$$

FLRW-like evolution

Local background evolves stochastically

$$\phi' = \pi + \xi_\phi \,, \qquad \pi' = -\left(3 - \frac{1}{2}\pi^2\right)\pi - \frac{V'}{H^2} + \xi_\phi$$

$$\left(3 - \frac{1}{2}\pi^2\right)H^2 = V$$

FLRW-like evolution with noise

Noise originates from quantum vacuum

Short-wavelength equation of motion:

$$\delta\phi_k'' = -\left(3 - \tfrac{1}{2}\pi^2\right)\delta\phi_k' \\ -\left[\tfrac{k^2}{a^2H^2} + \pi^2\left(3 - \tfrac{1}{2}\pi^2\right) + 2\pi\tfrac{V'(\phi)}{H^2} + \tfrac{V''(\phi)}{H^2}\right]\delta\phi_k$$

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with Bunch-Davies vacuum,

$$\delta\phi_k = \tfrac{1}{\sqrt{2ka}}\,, \qquad \delta(a\phi_k)' = -i\tfrac{k}{H}\delta\phi_k\,, \qquad k \gg aH$$

Noise originates from quantum vacuum

Noise from modes crossing k_σ; quantum randomness

$$\langle \xi_\phi(N)\xi_\phi(N')\rangle = \tfrac{1}{6\pi^2}\tfrac{\mathrm{d}k_\sigma^3}{\mathrm{d}N}|\delta\phi_{k_\sigma}(N)|^2\delta(N-N')\,,$$

$$\langle \xi_\pi(N)\xi_\pi(N')\rangle = \tfrac{1}{6\pi^2}\tfrac{\mathrm{d}k_\sigma^3}{\mathrm{d}N}|\delta\phi'_{k_\sigma}(N)|^2\delta(N-N')\,,$$

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$$\begin{aligned}\tfrac{1}{6\pi^2}\tfrac{\mathrm{d}k_\sigma^3}{\mathrm{d}N}|\delta\phi_{k_\sigma}(N)|^2 &= (1-\epsilon_1)\tfrac{k_\sigma^3}{2\pi^2}|\delta\phi_{k_\sigma}(N)|^2 \\ &\equiv (1-\epsilon_1)\mathcal{P}_{\phi,\sigma}(N)\end{aligned}$$

Comoving curvature perturbation

Linear level:

$\mathcal{R}_k = \delta\phi_k/\pi$

Comoving curvature perturbation

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$\mathcal{R}_k = \delta\phi_k/\pi$

Non-linear level:

$\mathcal{R} = \Delta N \equiv N - \langle N \rangle$

("ΔN formalism")

Comoving curvature perturbation freezes

Super-Hubble scales, $k \ll aH$:

$\mathcal{R}_k'' + (3 - \epsilon_1 + \epsilon_2)\mathcal{R}_k' = 0$

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$\mathcal{R}_k'' + (3 - \epsilon_1 + \epsilon_2)\mathcal{R}_k' = 0$

For $\epsilon_2 > \epsilon_1 - 3$, $\mathcal{R}$ freezes:

$\mathcal{R}_k' \to 0$

Solving for curvature perturbations

Evolve ϕ and $\delta\phi_k$ for many modes k

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Stop stochastic kicks at fixed $N = N_c$

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Evolve to a fixed $\phi = \phi_{\mathsf{final}}$

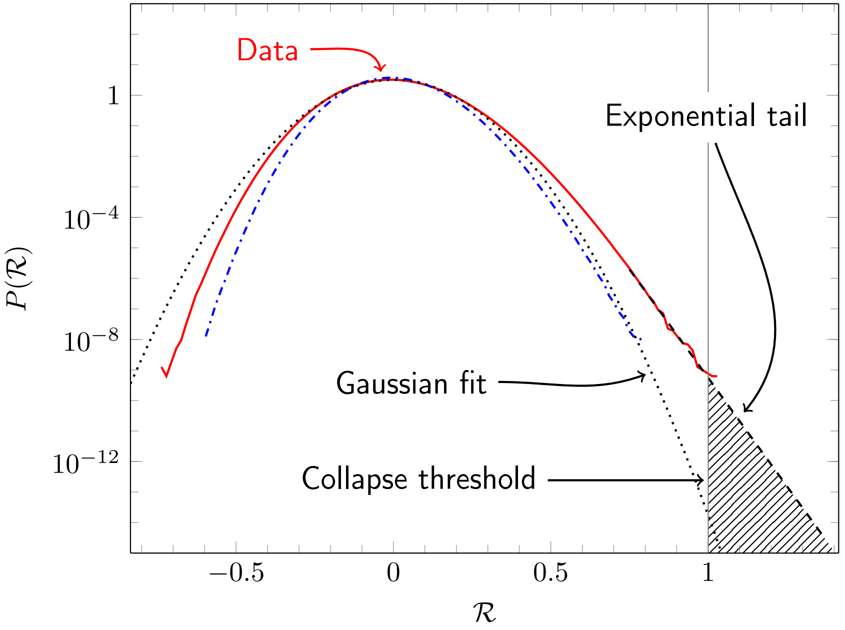
Solving for curvature perturbations

Evolve ϕ and $\delta\phi_k$ for many modes k

Stop stochastic kicks at fixed $N = N_c$

Evolve to a fixed $\phi = \phi_{\text{final}}$

Read off $\Delta N = \mathcal{R}$ (ΔN formalism)



Data
Exponential tail
Gaussian fit
Collapse threshold
1
10^{-4}
10^{-8}
10^{-12}
$P(\mathcal{R})$
-0.5
0
0.5
1
$\mathcal{R}$

History of stochastic inflation

Seminal work [Starobinsky 1986]

ΔN formalism [Fujita 2013]

Primordial black holes [Pattison 2017]

Exponential tails [Ezquiaga 2019]

Beyond de Sitter noise, with bakcreaction
[Figueroa 2020, 2021]

Development I:

Constraining motion to one dimension

Freezing aligns perturbations

$$\frac{\delta\phi_k'}{\delta\phi_k} = \frac{\pi'}{\pi} + \frac{\mathcal{R}_k'}{\mathcal{R}_k}$$

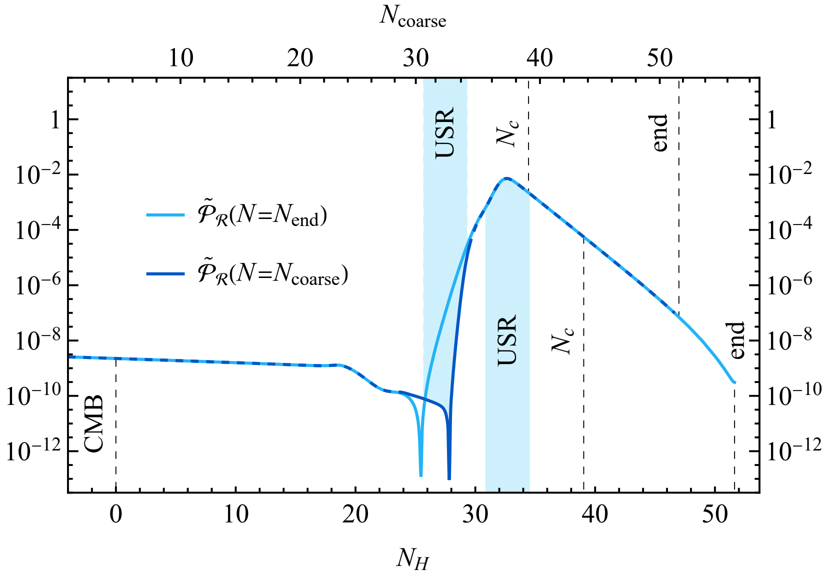
Perturbations align with the background on an attractor:

$\delta\phi_k \to c\,\pi$ for $\mathcal{R}_k'/\mathcal{R}_k \to 0$

Squeezing transfers this to noise:

$\xi_\pi = \xi_\phi \frac{\delta\phi_k'}{\delta\phi_k}\Big|_{k=k_\sigma}$

Perturbations frozen when giving kicks



Motion along classical trajectory

Classical trajectory:

$N = \tilde{N}$, $\phi = \tilde{\phi}$, $\pi = \tilde{\pi}$, $\epsilon_n = \tilde{\epsilon}_n$

Stochastic equation:

$\phi' = \tilde{\pi}(\phi) + \xi_\phi$

Motion along classical trajectory

Classical trajectory:

$N = \tilde{N}$, $\phi = \tilde{\phi}$, $\pi = \tilde{\pi}$, $\epsilon_n = \tilde{\epsilon}_n$

Stochastic equation:

$$\mathrm{d}\phi/\mathrm{d}N = \tilde{\pi}(\phi) + \sqrt{(1-\tilde{\epsilon}_1)\mathcal{P}_{\phi,\sigma}/\mathrm{d}N}\,\hat{\xi}_i\,, \qquad \left\langle \hat{\xi}_i \hat{\xi}_j \right\rangle = \delta_{ij}$$

'Constrained stochastic inflation'

Development II:

Classical number of e-folds as a stochastic variable

Changing from ϕ to $\tilde{N}$

A change of variables:

$$\mathrm{d}\phi = \tilde{\pi}(\tilde{N})\mathrm{d}\tilde{N}$$

Equation becomes:

$$\mathrm{d}\tilde{N} = \mathrm{d}N + \sqrt{\left[1 - \tilde{\epsilon}_1(\tilde{N})\right] \frac{\mathcal{P}_{\phi,\sigma}}{2\tilde{\epsilon}_1(\tilde{N})} \mathrm{d}N}\, \hat{\xi}_i$$

Connection to ΔN formalism

At any moment:

$\Delta N = N - \tilde{N}$

This grows from 0 to its final value during stochastic evolution.

Gaussian limit: a standard result

Limit $\Delta N \ll 1$: $N \approx \tilde{N}$ with independent kicks,

$$\mathrm{d}\tilde{N} \approx \mathrm{d}N + \sqrt{[1-\tilde{\epsilon}_1(N)]\tilde{\mathcal{P}}_{\mathcal{R},\sigma}(N)\mathrm{d}N}\,\hat{\xi}_i$$

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ΔN distribution is Gaussian, with variance

$$\langle \Delta N^2 \rangle = \sum_{i=1}^{n} [1-\tilde{\epsilon}_1(N_i)]\tilde{\mathcal{P}}_{\mathcal{R},\sigma}(N_i)\mathrm{d}N$$

$$\xrightarrow[\epsilon_1 \ll 1]{\mathrm{d}N \to 0} \int_{N_{\mathsf{ini}}}^{N_{\mathsf{c}}} \tilde{\mathcal{P}}_{\mathcal{R},\sigma}(N)\mathrm{d}N \approx \int_{k_{\mathsf{ini}}}^{k_{\mathsf{c}}} \tilde{\mathcal{P}}_{\mathcal{R}}(k)\,\mathrm{d}\ln k$$

Development III:

Perturbation evolution is independent of stochastic noise

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Perturbation evolution is independent of stochastic noise ...during constant-roll

Frozen perturbations behave predictably

Frozen perturbations: $\delta\phi_k \sim \sqrt{\epsilon_1}$

$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}N} \ln \delta\phi_k = \frac{1}{2}\epsilon_2$

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$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}N} \ln \delta\phi_k = \frac{1}{2}\epsilon_2$

Constant roll: $\epsilon_2 =$ const

Note: this is a constant everywhere in the CR phase!

Match from pre-computed perturbations

Compute perturbations on the classical background:

$$\mathcal{P}_{\phi,\sigma} = \tilde{\mathcal{P}}_{\phi,\sigma}(N)$$

Equations become:

$$\mathrm{d}\tilde{N} = \mathrm{d}N + \sqrt{\tilde{P}(N,\tilde{N})\mathrm{d}N}\,\hat{\xi}_i\,,$$

$$\tilde{P}(N,\tilde{N}) \equiv \frac{\tilde{\mathcal{P}}_{\phi,\sigma}(N)}{2\tilde{E}_1(\tilde{N})}\,, \quad \tilde{E}_1(\tilde{N}) \equiv \frac{\tilde{\epsilon}_1(\tilde{N})}{1-\tilde{\epsilon}_1(\tilde{N})}$$

Development IV:

Importance sampling

Development IV:

Importance sampling ...along pre-computed paths

Direct vs importance sampling

Direct sampling: solve the equation by pulling $\hat{\xi}_i$ randomly from Gaussian distributions

- A lot of effort to access the tail of $p(\Delta N)$

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- A lot of effort to access the tail of $p(\Delta N)$

Importance sampling: write $\hat{\xi}_i = \bar{\xi}_i + \delta\xi_i$, and

$$
\begin{aligned}
p &= \tfrac{1}{(2\pi)^{n/2}} \exp\Big[-\tfrac{1}{2}\textstyle\sum_i \hat{\xi}_i^2\Big] \\
&= \exp\big[-\tfrac{1}{2}\textstyle\sum_i \big(\bar{\xi}_i^2 + 2\bar{\xi}_i\delta\xi_i\big)\big] \times \exp\big[-\tfrac{1}{2}\textstyle\sum_i \delta\xi_i^2\big]
\end{aligned}
$$

Pull $\delta\xi_i$ randomly from Gaussian distributions; weight by the prefactor!

Importance sampling around what?

Statistics converge faster around a particular ΔN for a particularly chosen $\bar{\xi}_i$

Choose $\bar{\xi}_i$ to follow the 'most probable path'

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$S_\xi = \frac{1}{2}\sum_i \hat{\xi}_i^2$

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$$\delta S_\xi = 0$$

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$$\delta S_\xi = 0 \Rightarrow \tilde{N}'' - \frac{\tilde{E}_1'(\tilde{N})}{2\tilde{E}_1(\tilde{N})}\left(1-\tilde{N}'^2\right) + \frac{\tilde{\mathcal{P}}'_{\phi,\sigma}(N)}{\tilde{\mathcal{P}}_{\phi,\sigma}(N)}\left(1-\tilde{N}'\right) = 0$$

Most probable paths

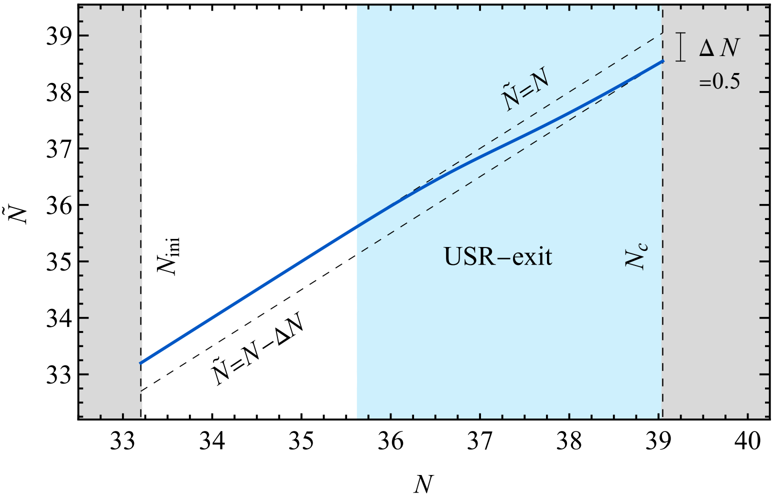
Solve the most probable path from

$$\tilde{N}'' - \frac{\tilde{E}_1'(\tilde{N})}{2\tilde{E}_1(\tilde{N})}\left(1 - \tilde{N}'^2\right) + \frac{\tilde{\mathcal{P}}'_{\phi,\sigma}(N)}{\tilde{\mathcal{P}}_{\phi,\sigma}(N)}\left(1 - \tilde{N}'\right) = 0$$

Boundary conditions: $\tilde{N} = N$ at N_{ini}; $\tilde{N} = N - \Delta N$ at N_c

Such a path *maximizes the probability density for a fixed* ΔN

Most probable paths



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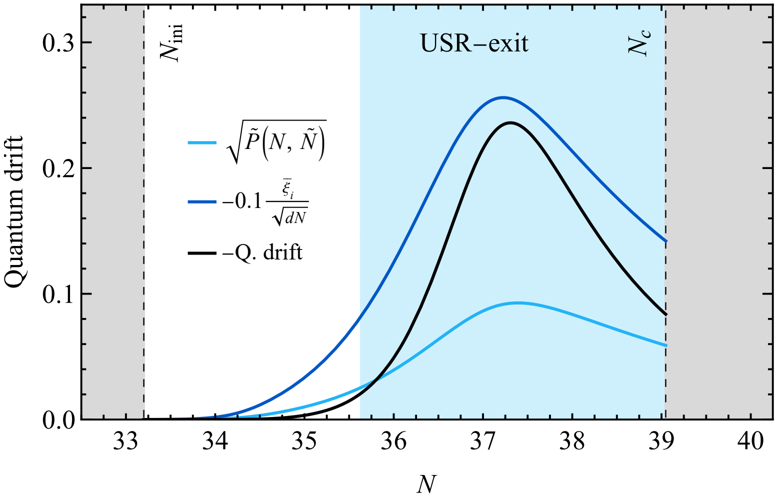


Analytical estimate

Estimate:

$$p(\Delta N)\,\mathrm{d}(\Delta N) = \int_{D(\Delta N)} \frac{\mathrm{d}^n \hat{\xi}_i}{(2\pi)^{n/2}} \exp\left[-\frac{1}{2}\sum_i \hat{\xi}_i^2\right]$$

Analytical estimate

Estimate:

$$p(\Delta N)\,\mathrm{d}(\Delta N) \approx \frac{\sqrt{\sum_i \bar{\xi}_i^2}}{|\Delta N|}\frac{\mathrm{d}(\Delta N)}{\sqrt{2\pi}}\exp\left[-\tfrac{1}{2}\sum_i \bar{\xi}_i^2\right]$$

Importance sampling = computing the volume factor numerically

Numerics

Compare two cases:

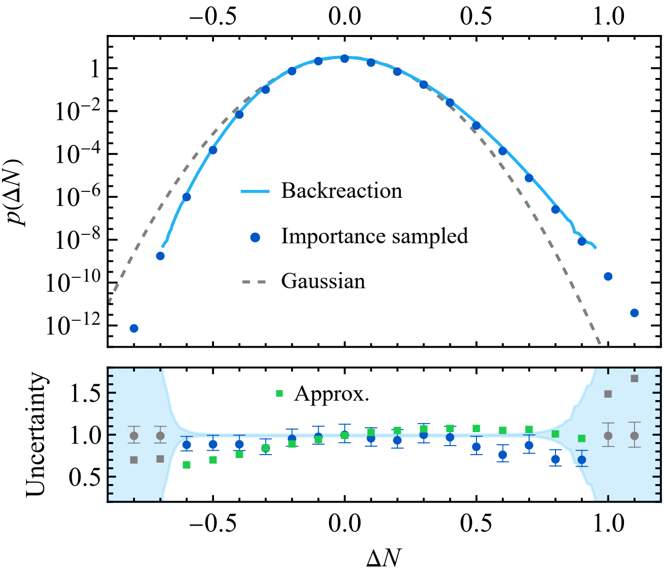
Backreaction computation from [Figueroa 2020, 2021]

- 1024×10^8 runs, $p(\Delta N)$ resolved continuously from -0.69 to 0.95

Constrained importance sampling

- 26×10^4 runs, $p(\Delta N)$ resolved from -1 to 1.5 in steps of 0.1

Numerics



−0.5
0.0
0.5
1.0
1
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
$p(\Delta N)$
Backreaction
Importance sampled
Gaussian
Approx.
1.5
1.0
0.5
Uncertainty
−0.5
0.0
0.5
1.0
ΔN

Numerics

Backreaction: million CPU hours

Constrained importance sampling: 2s

Time saving of factor 10^9

Summary

Direct sampling: see non-Gaussian tail with a million CPU hours or more

A number of developments:

- Frozen noise constrains motion to one dimension
- Use classical number of e-folds as a stochastic variable
- Perturbations don't depend on stochasticity in constant roll
- Importance sampling around most probable paths

Get same result within seconds

Conclusions

Non-Gaussianity is important for inflationary PBH formation

Stochastic computation beyond de Sitter noise is needed

A number of analytical insights can simplify the computation

Goal: make accurate PBH computations accesible to everyone

Thank you!

Gaussian limit

[present Gaussian limit... IF time]

Exponential tail

[present exponential tail... IF time]