Primordial black holes and stochastic inflation

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in collaboration with D. Figueroa, S. Raatikainen, S. Räsänen

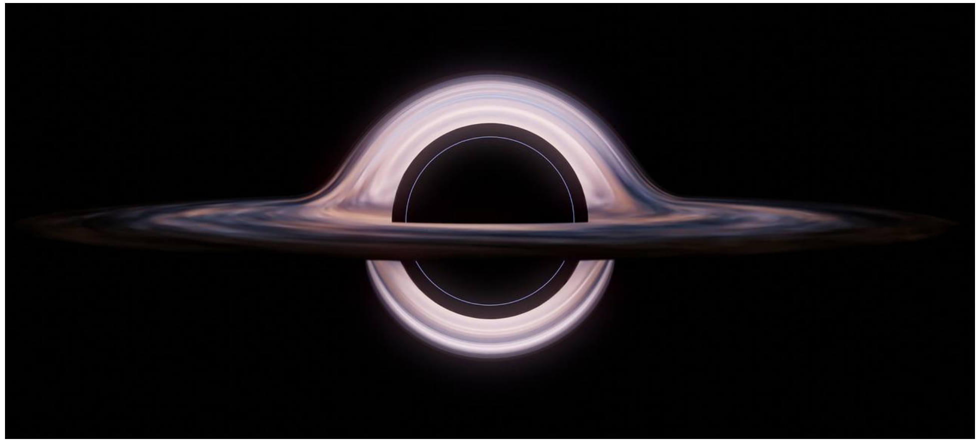
Why primordial black holes (PBHs)?

Black holes formed in early Universe

- Carry information of conditions there (small-scale perturbations)
- Any mass (Hawking evaporation?)

Applications in cosmology

- Dark matter candidate
- Seeds of supermassive black holes



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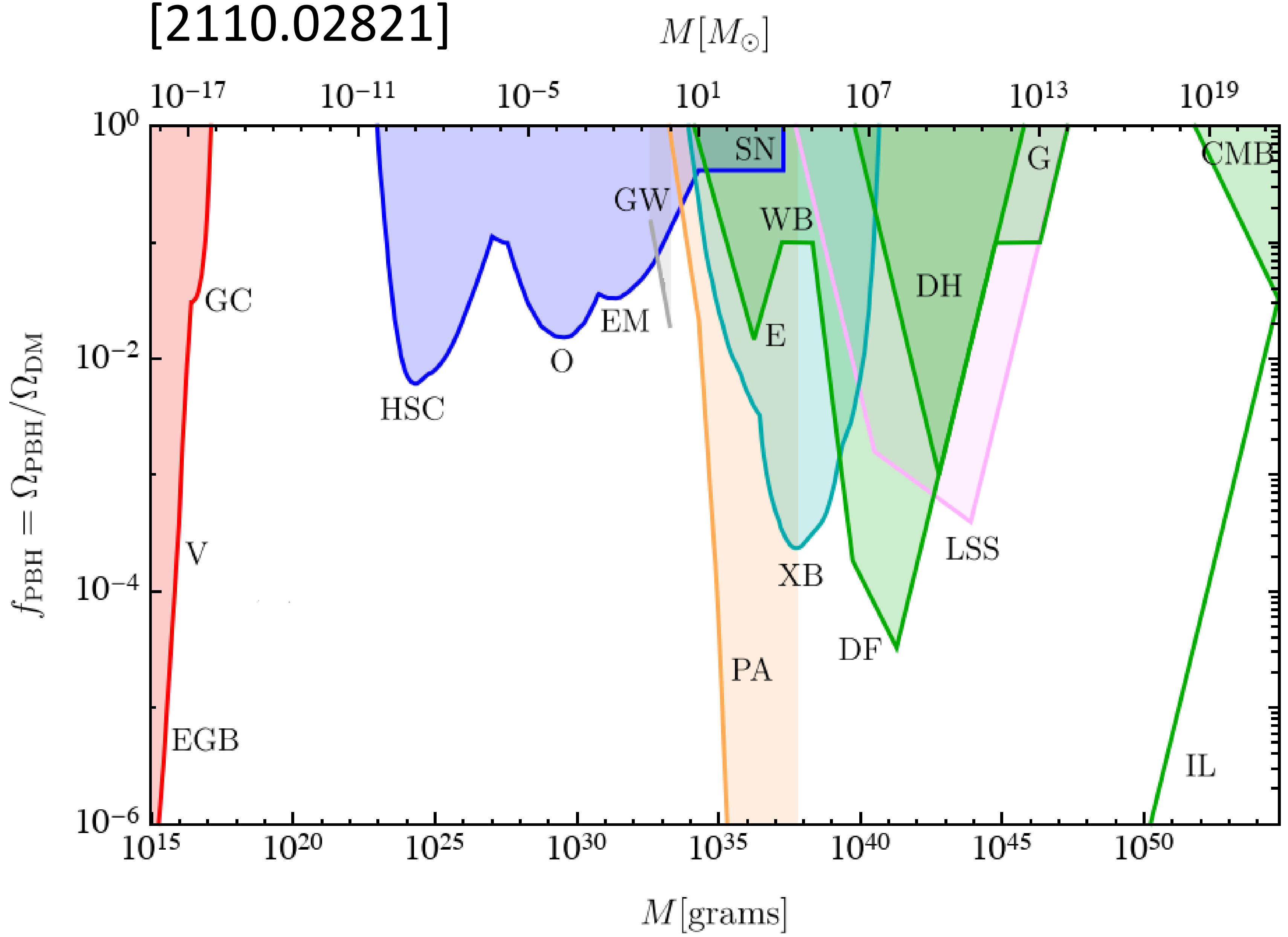


M [M⊙]
10−17
10−11
10−5
10¹
10⁷
10¹³
10¹⁹
10⁰
10−2
10−4
10−6
f_PBH = Ω_PBH/Ω_DM
GC
V
EGB
HSC
O
EM
GW
SN
WB
E
XB
PA
DF
DH
G
LSS
CMB
IL
10¹⁵
10²⁰
10²⁵
10³⁰
10³⁵
10⁴⁰
10⁴⁵
10⁵⁰
M [grams]

Renewed interest since GW detection

Gravitational wave signals:

- Black hole mergers
- Stochastic scalar-induced gravitational waves

Matching stochastic GW signal to PBH statistics?

- GWs: sourced by typical scalar perturbations
- PBHs: sourced by extreme scalar perturbations

Origins of primordial black holes

Cosmological phase transitions

Cosmic strings

Primordial perturbations: cosmic inflation

Origins of primordial black holes

Cosmological phase transitions

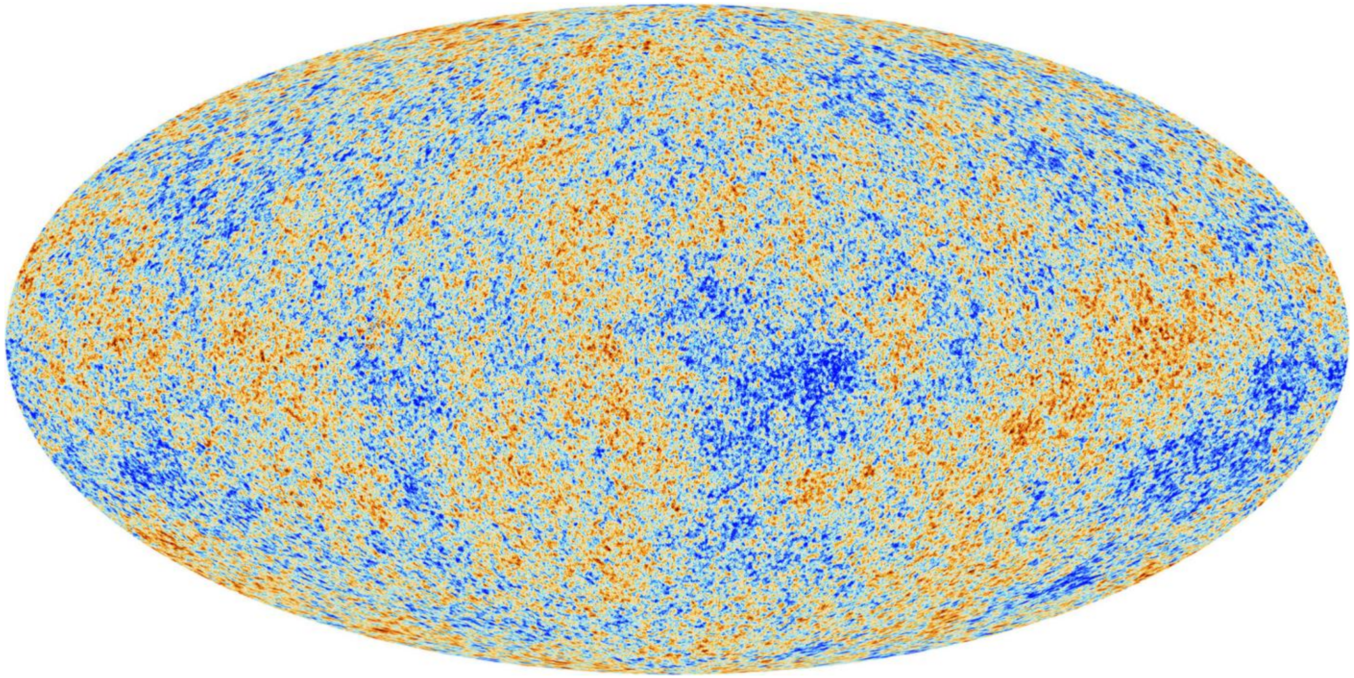
Cosmic strings

Primordial perturbations: cosmic inflation

Black holes from primordial perturbations

Cosmic inflation: quantum fluctuations

Later: strongest collapse into black holes



I. (Semi-)inflection point inflation

II. Stochastic inflation

III. Black hole statistics

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Single-field inflation is simple

Action:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - V(\varphi)\right]$$

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Background equations of motion:

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0\,, \quad 3H^2 = \frac{1}{2}\dot{\varphi}^2 + V(\varphi)$$

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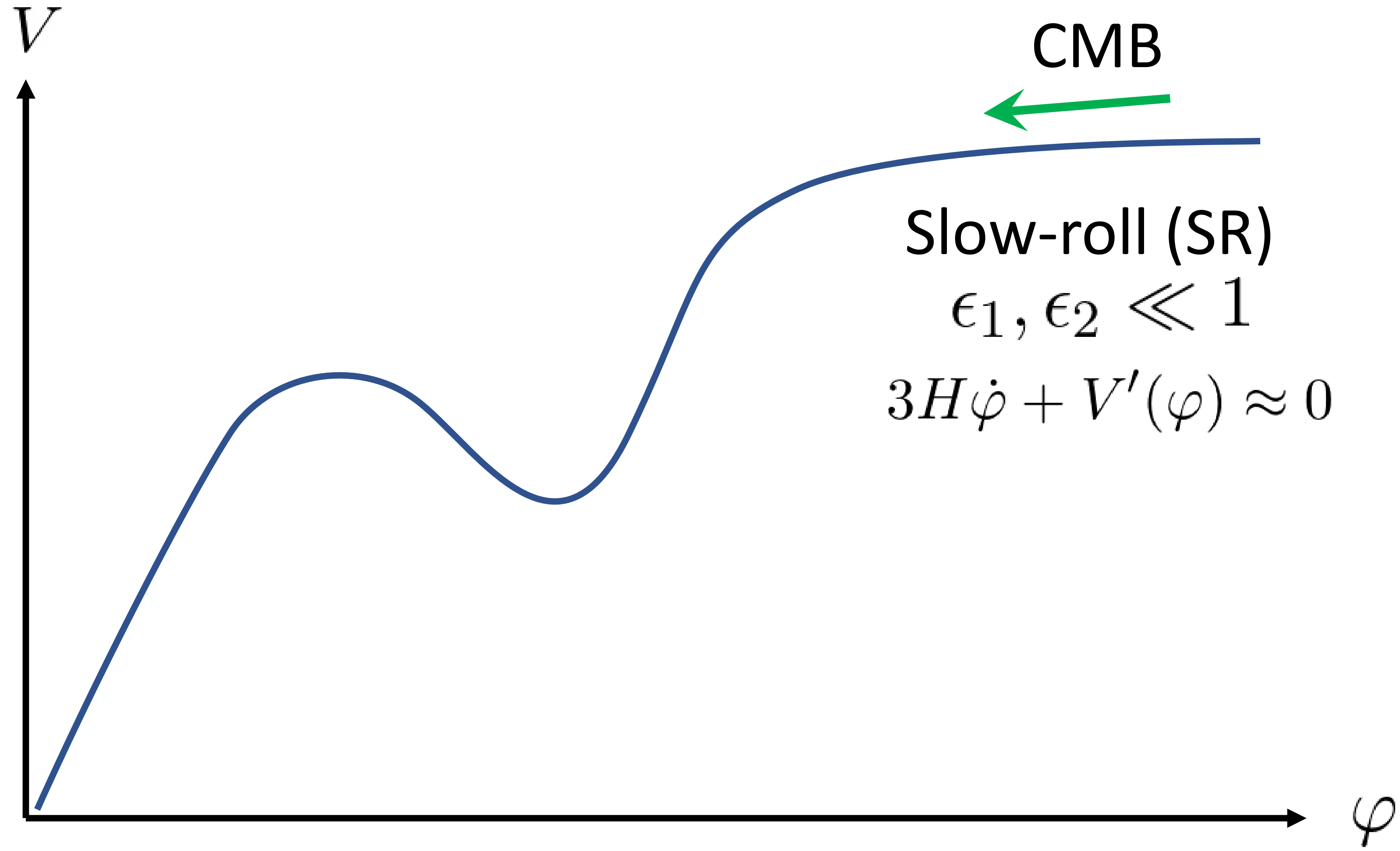
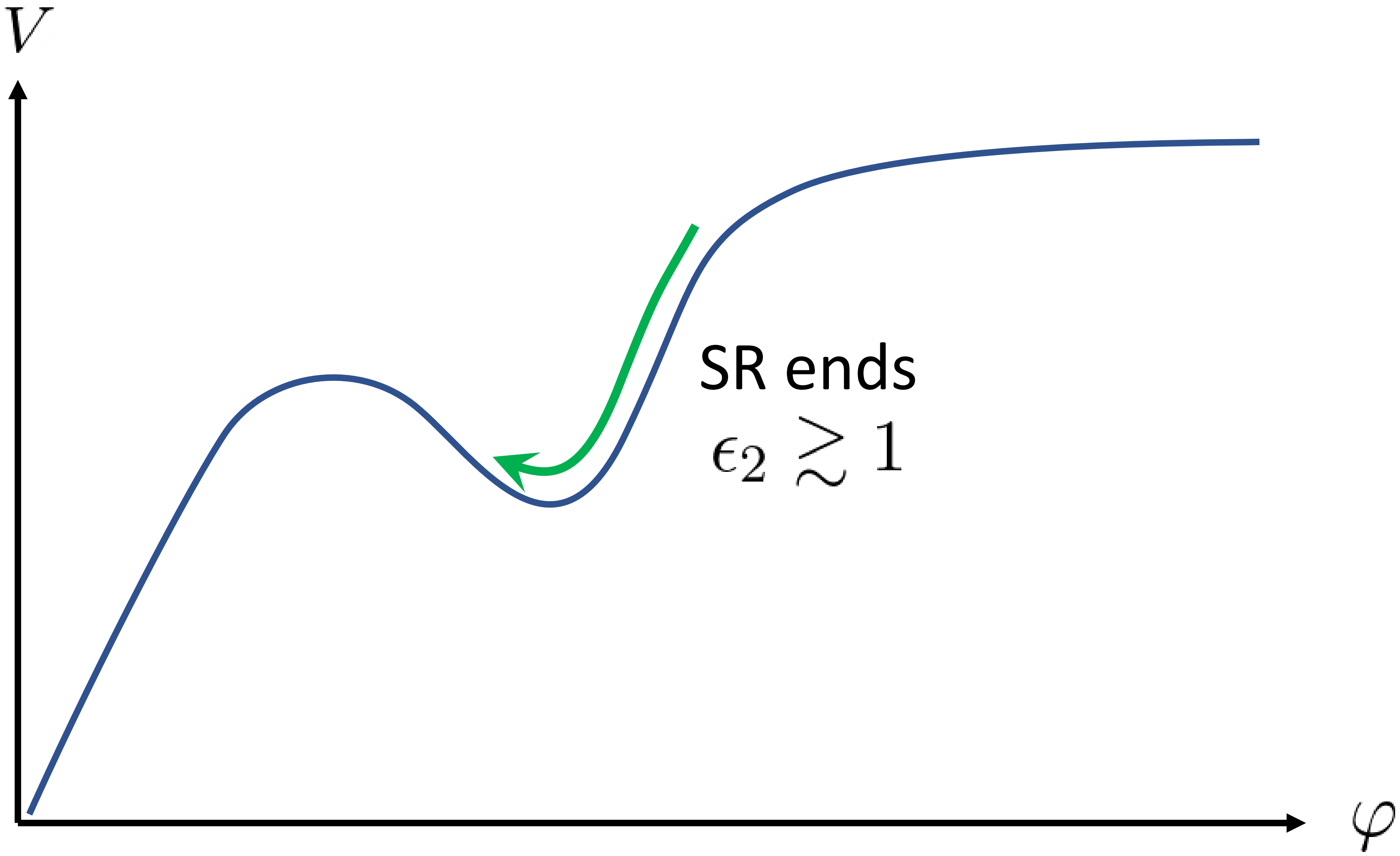
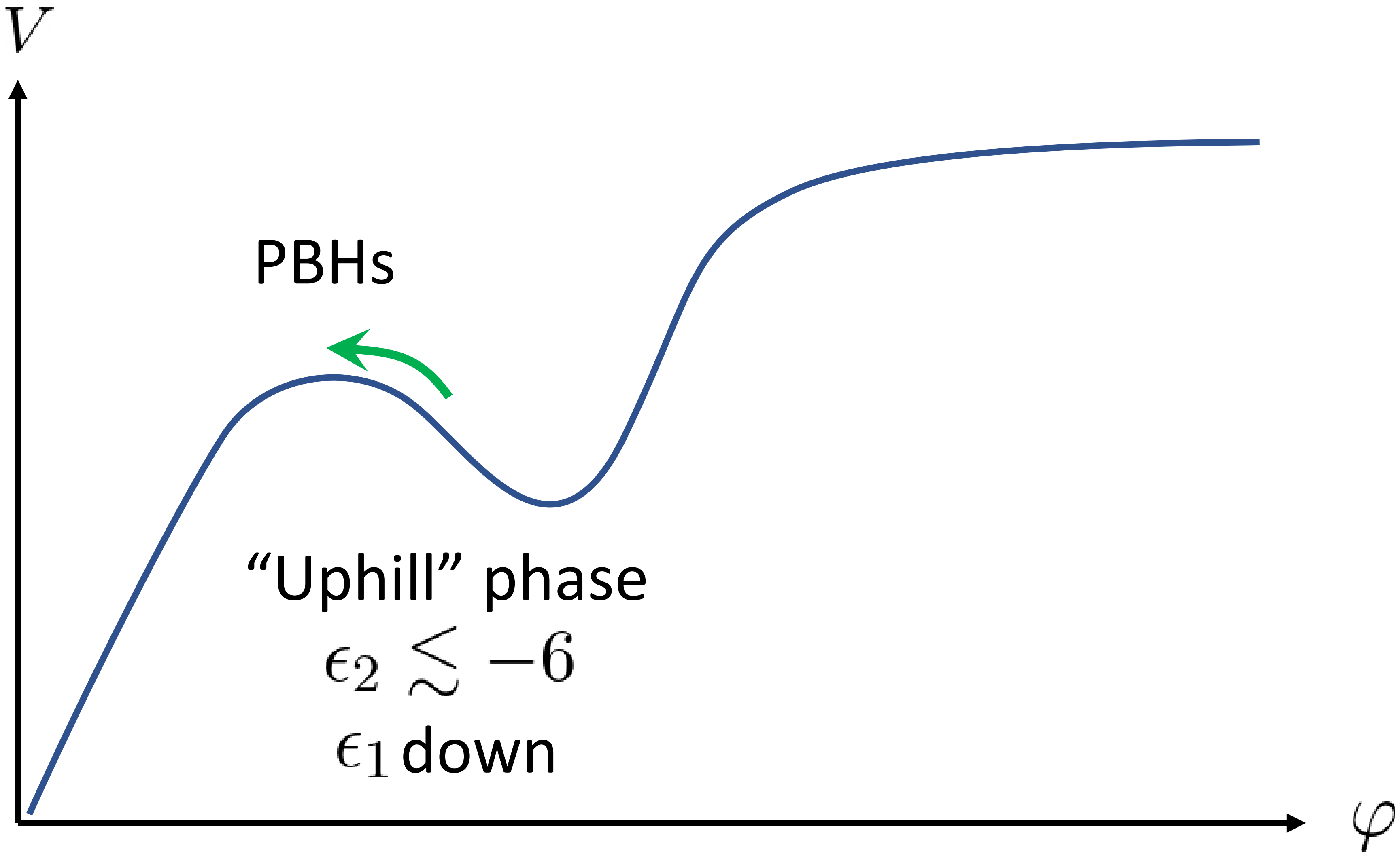
Slow-roll parameters:

$$\epsilon_1 \equiv -\partial_N \ln H\,, \quad \epsilon_2 \equiv \partial_N \ln \epsilon_1$$

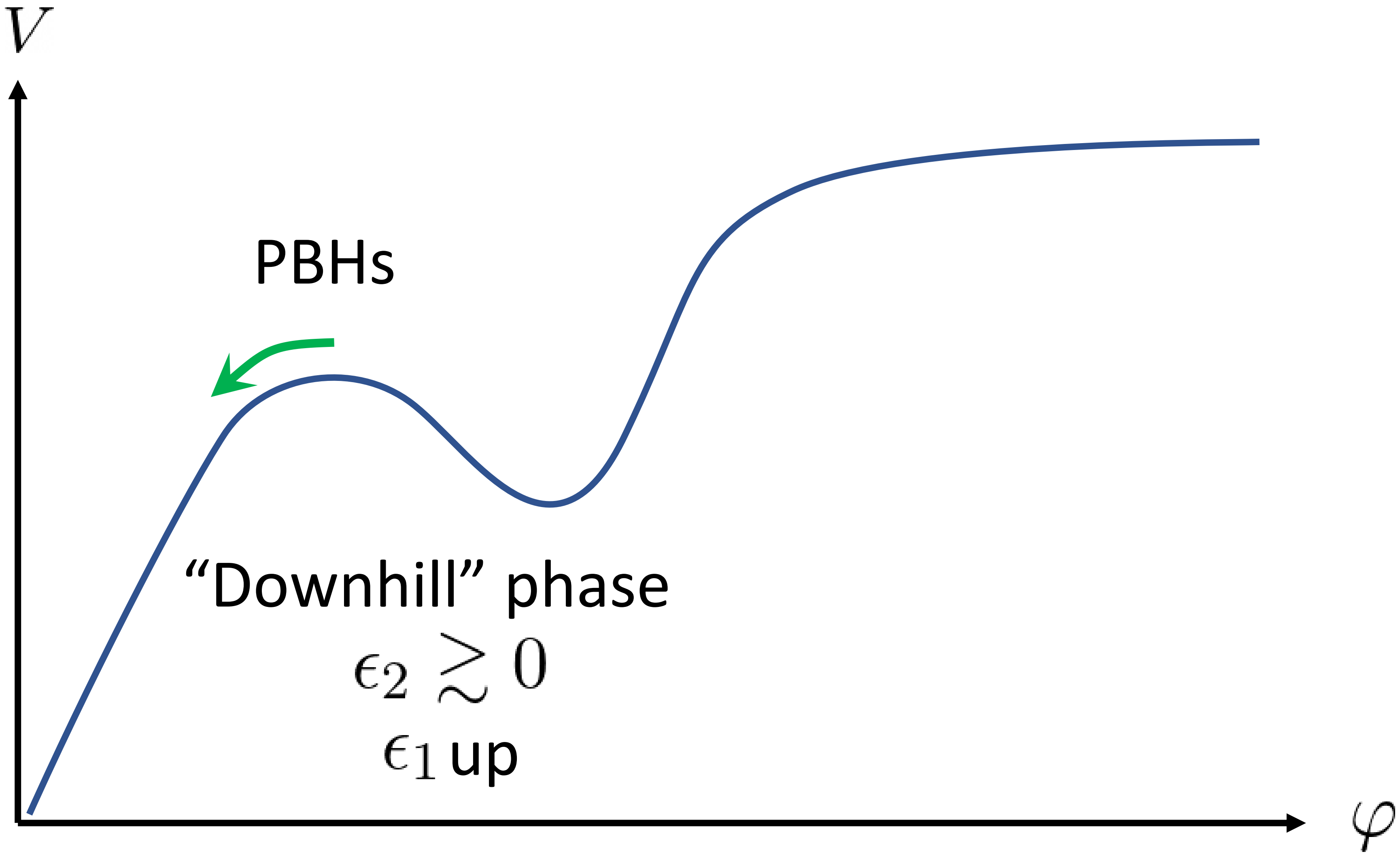

V
CMB
Slow-roll (SR)
$\epsilon_1, \epsilon_2 \ll 1$
$3H\dot{\varphi} + V'(\varphi) \approx 0$
φ



V
SR ends
$\epsilon_2 \gtrsim 1$
φ



V
PBHs
"Uphill" phase
$\epsilon_2 \lesssim -6$
ϵ_1 down
φ



V
PBHs
"Downhill" phase
$\epsilon_2 \gtrsim 0$
ϵ_1 up
φ

Linear perturbations grow near feature

Comoving curvature perturbation $\mathcal{R} = \frac{\delta\varphi}{\sqrt{2\epsilon_1}}$

$$\ddot{\mathcal{R}}_k + H(3+\epsilon_2)\dot{\mathcal{R}}_k + \frac{k^2}{a^2}\mathcal{R}_k = 0$$

Linear perturbations grow near feature

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Vacuum initial conditions:

$$\mathcal{R}_k = \frac{1}{2a\sqrt{k\epsilon_1}} e^{ik/(aH)}$$

Late times:

$$\mathcal{R}_k \to \text{const. if } \epsilon_2 > -3$$

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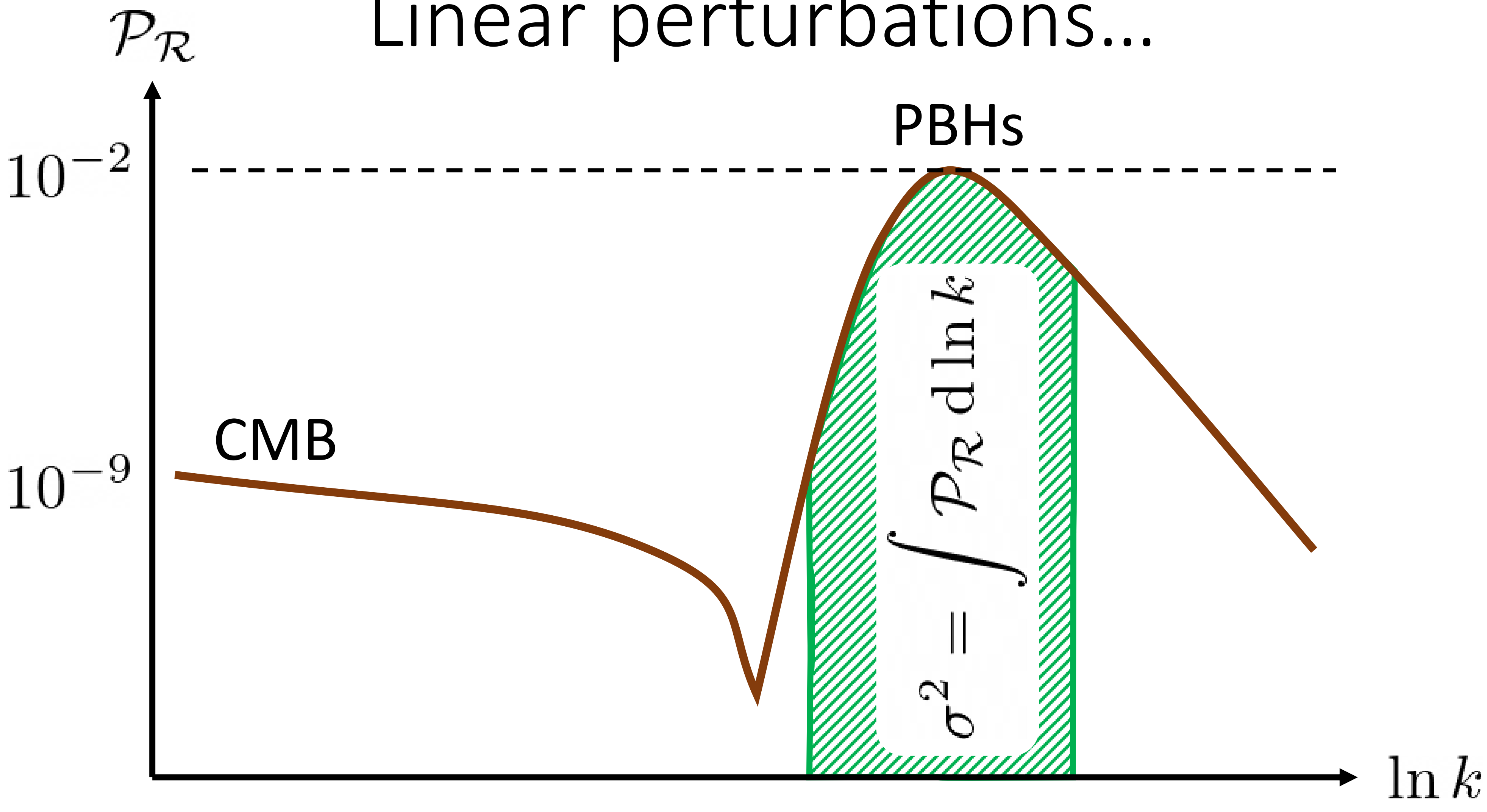
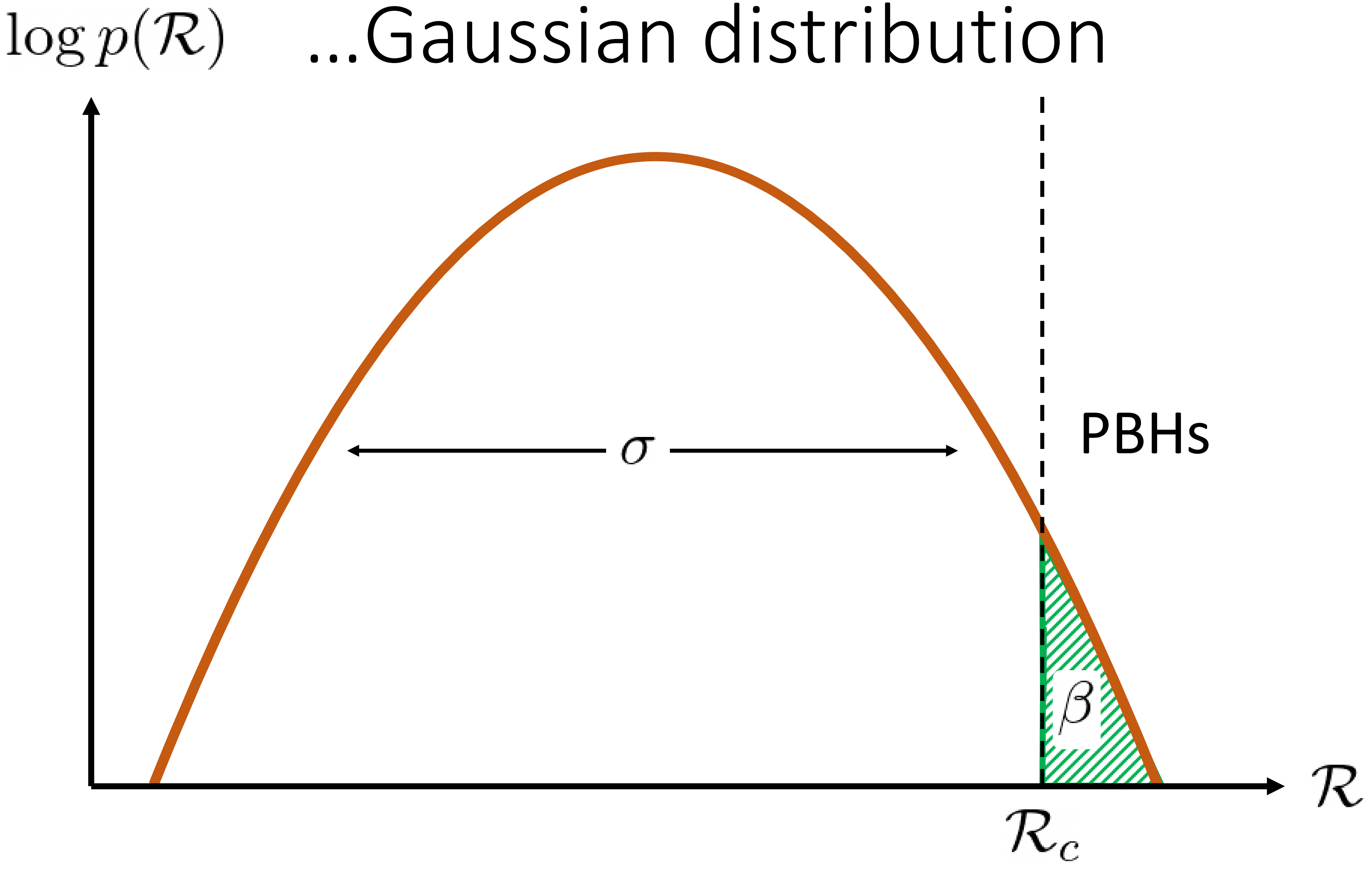
Define power spectrum: $\mathcal{P}_{\mathcal{R}}(k) \equiv \frac{k^3}{2\pi^2}|\mathcal{R}_k|^2$

Linear perturbations...



…Gaussian distribution



Why this picture is wrong

$\mathcal{R}$ is not the correct statistic for PBH formation

Perturbations in the tail are not Gaussian

I. (Semi-)inflection point inflation

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Approximations in two regimes

Sub-Hubble scales:

Linear perturbation theory good; neglect mode couplings

$$\delta\ddot{\varphi}_k + 3H\delta\dot{\varphi}_k + H^2\left(\frac{k^2}{a^2H^2} - \frac{3}{2}\epsilon_2 + \frac{1}{2}\epsilon_1\epsilon_2 - \frac{1}{4}\epsilon_2^2 - \frac{1}{2}\epsilon_2\epsilon_3\right)\delta\varphi_k = 0$$

Super-Hubble scales:

Local FLRW equations good; neglect gradient terms

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0$$

Approximations in two regimes

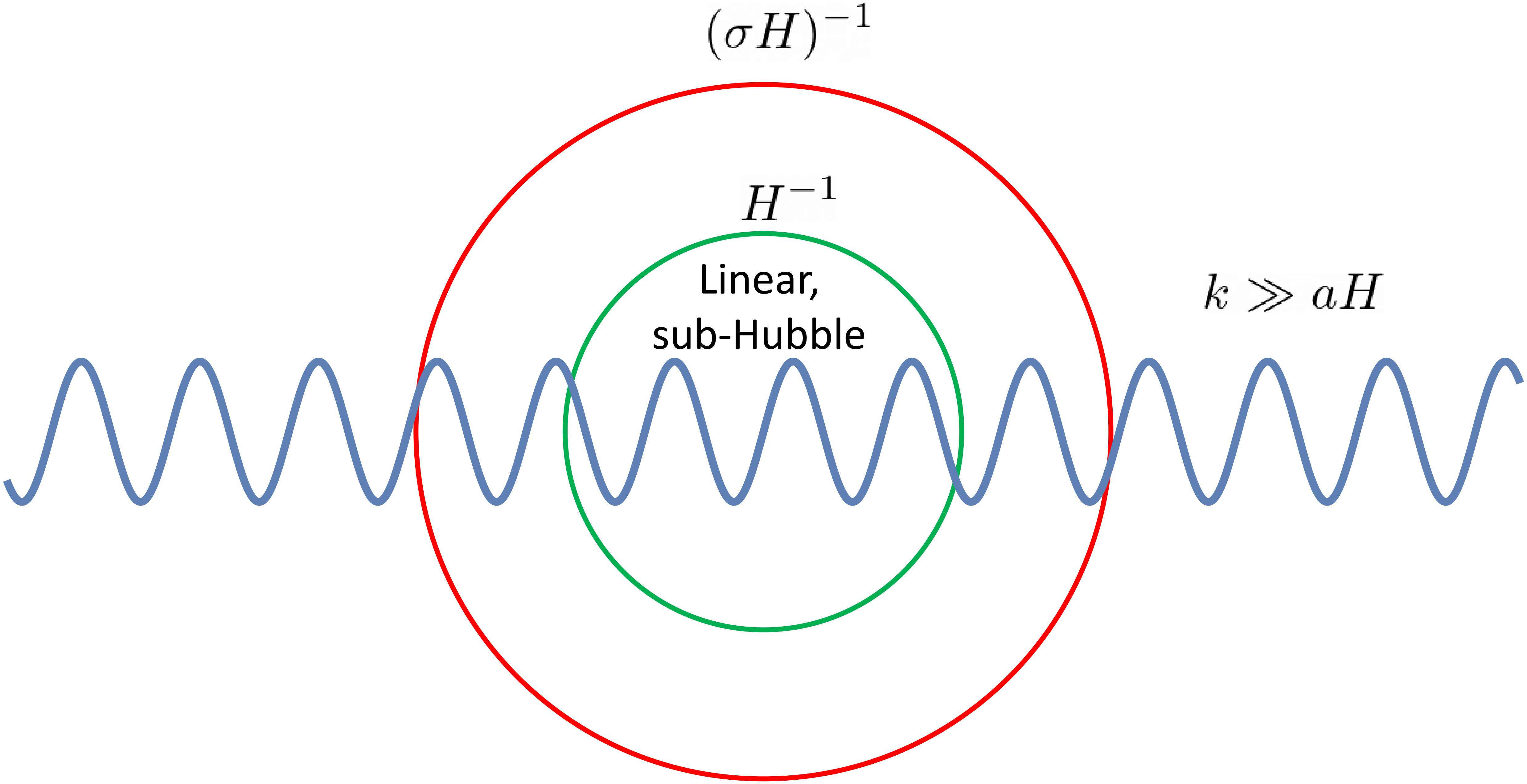
Inflaton field: $\varphi = \phi + \delta\phi$

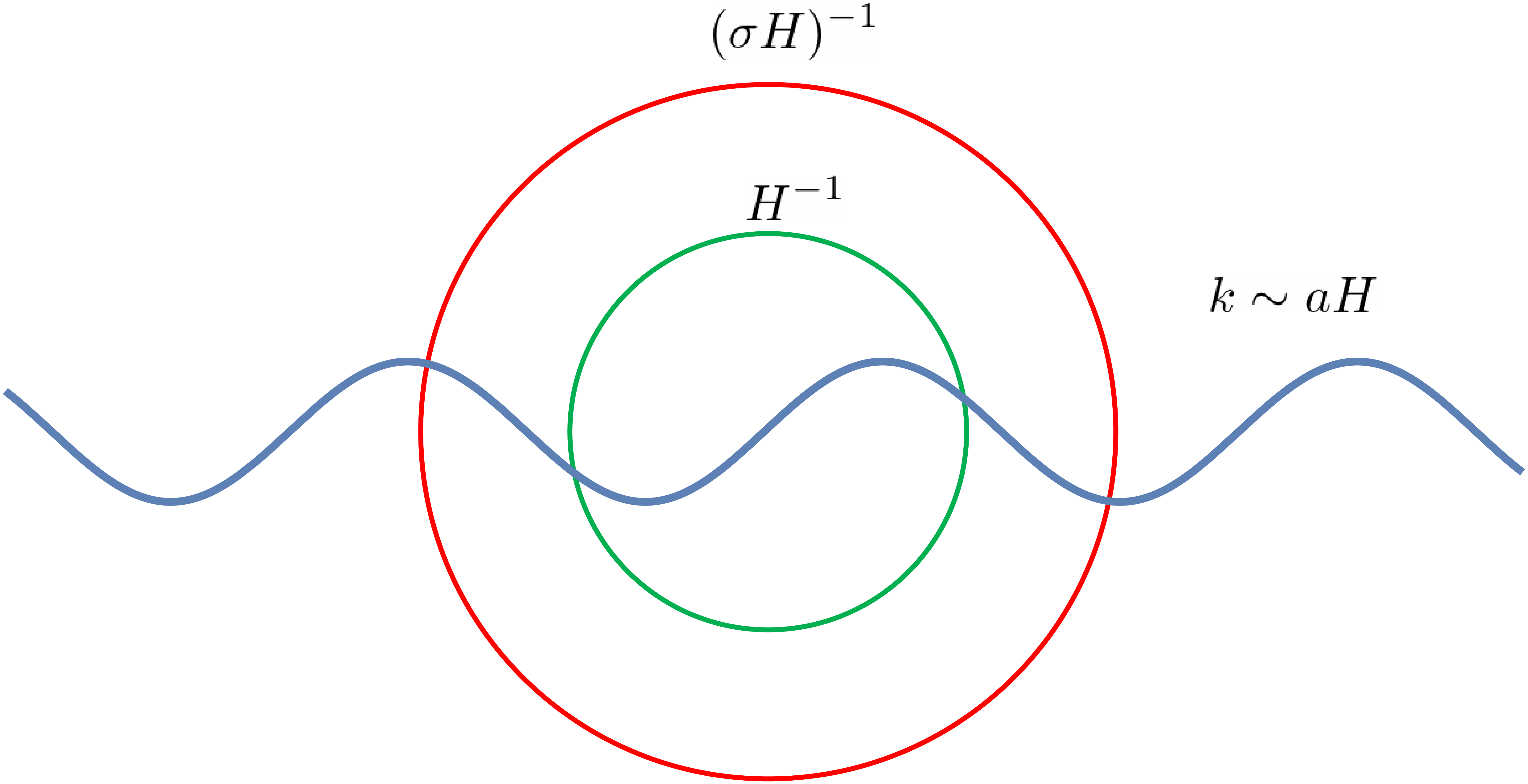
Coarse-grained:
FLRW

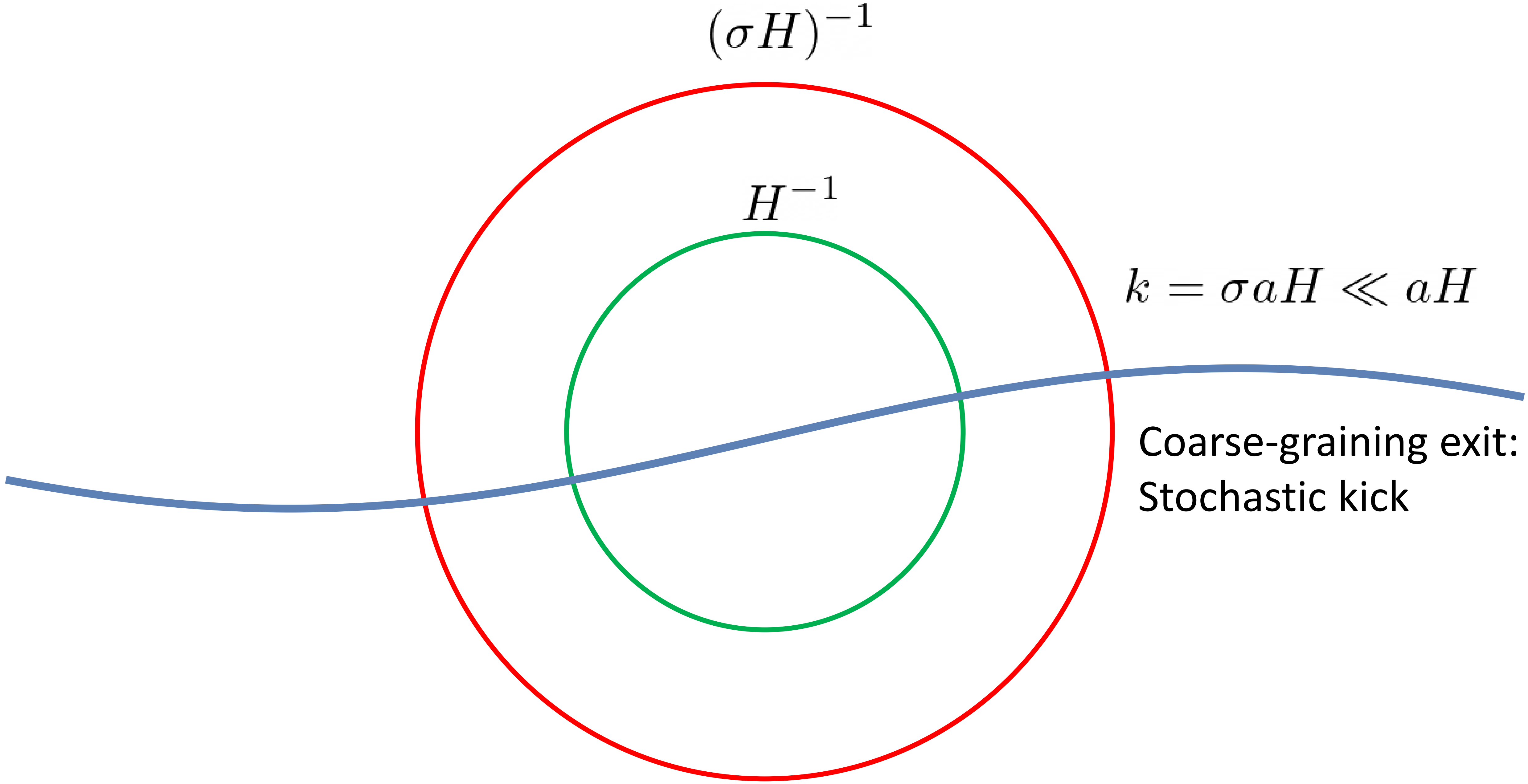
Short-wavelength:
linear perturbation theory

$$\phi \equiv \int_{k<k_\sigma} \frac{\mathrm{d}^3k}{(2\pi)^{2/3}} \varphi_k(N) e^{-i\vec{k}\cdot\vec{x}} \qquad \delta\phi \equiv \int_{k>k_\sigma} \frac{\mathrm{d}^3k}{(2\pi)^{2/3}} \varphi_k(N) e^{-i\vec{k}\cdot\vec{x}}$$

Patched together at the coarse-graining scale $k = k_\sigma \equiv \sigma a H$


$(\sigma H)^{-1}$
H^{-1}
Linear,
sub-Hubble
$k \gg aH$


$(\sigma H)^{-1}$
H^{-1}
$k \sim aH$



$(\sigma H)^{-1}$
H^{-1}
$k = \sigma a H \ll aH$
Coarse-graining exit:
Stochastic kick

Stochastic inflation

$$\phi' = \pi + \xi_\phi\,, \quad \pi' = -\left(3 - \frac{1}{2}\pi^2\right)\pi - \frac{V'(\phi)}{H^2} + \xi_\pi\,, \quad H^2 = \frac{V(\phi)}{3 - \frac{1}{2}\pi^2}$$

$$\delta\phi_k'' = -(3 - \frac{1}{2}\pi^2)\delta\phi_k' - \left[\frac{k^2}{a^2H^2} + \pi^2(3 - \frac{1}{2}\pi^2) + 2\pi\frac{V'(\phi)}{H^2} + \frac{V''(\phi)}{H^2}\right]\delta\phi_k$$

$$\langle\xi_\phi(N)\xi_\phi(N')\rangle = \frac{1}{6\pi^2}\frac{\mathrm{d}k_\sigma^3}{\mathrm{d}N}|\delta\phi_{k_\sigma}(N)|^2\delta(N - N')$$

$$\langle\xi_\pi(N)\xi_\pi(N')\rangle = \frac{1}{6\pi^2}\frac{\mathrm{d}k_\sigma^3}{\mathrm{d}N}|\delta\phi'_{k_\sigma}(N)|^2\delta(N - N')$$

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$$\mathcal{R}_{<k} = \Delta N = N - \bar{N}$$

ΔN formalism

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)e^{2\zeta(x,t)}\mathrm{d}x^2$$

$$\Delta N \equiv N - \bar{N} = \mathcal{R} = \zeta$$

Stochastic ΔN formalism:

- solve stochastic system many times; include kicks up to scale k
- collect N on each run
- build statistics for coarse-grained curvature perturbation $\mathcal{R}_{<k}$

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$$\phi' = \pi + \xi_\phi\,, \quad \pi' = -\left(3 - \frac{1}{2}\pi^2\right)\pi - \frac{V'(\phi)}{H^2} + \xi_\pi\,, \quad H^2 = \frac{V(\phi)}{3 - \frac{1}{2}\pi^2}$$

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Stochastic inflation

$$\phi' = \pi + \xi_\phi \,, \quad \pi' = -\left(3 - \frac{1}{2}\pi^2\right)\pi - \frac{V'}{\cdots}$$ [illegible] $$H^2 = \frac{V(\phi)}{3 - \frac{1}{2}\pi^2}$$

$$\delta\phi_k'' = -(3 - \frac{1}{2}\pi^2)\delta\phi_k' - \left[\frac{k^2}{\cdots}\right.$$ [illegible] $$\left.\cdots) + 2\pi\frac{V'(\phi)}{H^2} + \frac{V''(\phi)}{H^2}\right]\delta\phi_k$$

$$\langle\xi_\phi(N)\xi_\phi(N')\rangle = \frac{1}{\cdots}$$ [illegible] $$\delta(N - N')$$

$$\langle\xi_\pi(N)\xi_\pi(N'$$ [illegible] $$\cdots_{k_\sigma}(N)|^2\delta(N - N')$$

$$\langle\xi_\phi(N)\xi_\pi(N$$ [illegible] $$\frac{\cdots_\sigma^3}{\mathrm{d}N}\delta\phi_{k_\sigma}(N)\delta\phi_{k_\sigma}'^*(N)\delta(N - N')$$

$$\mathcal{R}_{<k} = \Delta N = N - \bar{N}$$

COMPLICATED

How to move forward?

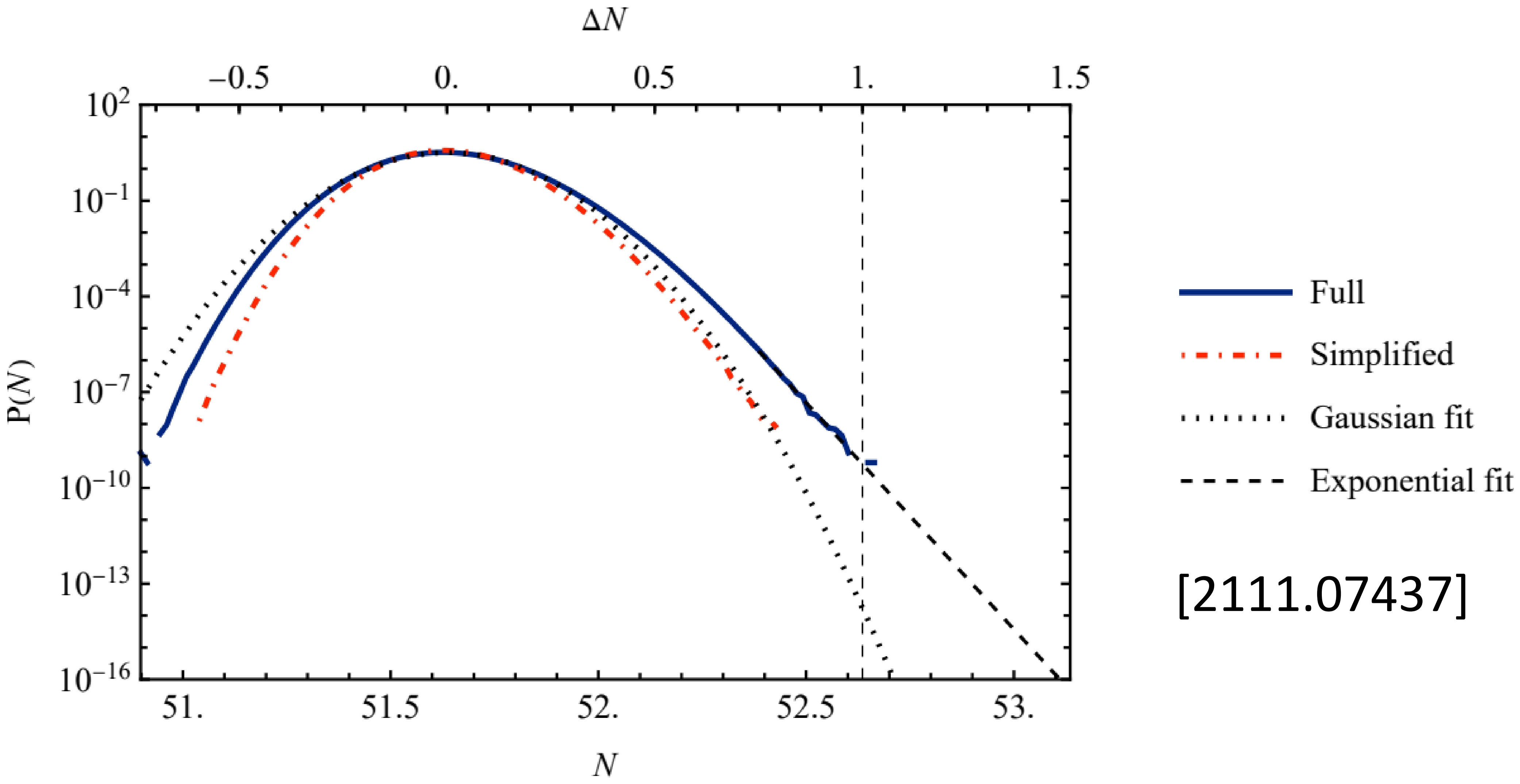
Analytical approximations?

$$\langle \xi_\phi(N)\xi_\phi(N')\rangle \approx \frac{H^2}{4\pi^2}\delta(N-N')$$

Full numerical computations?

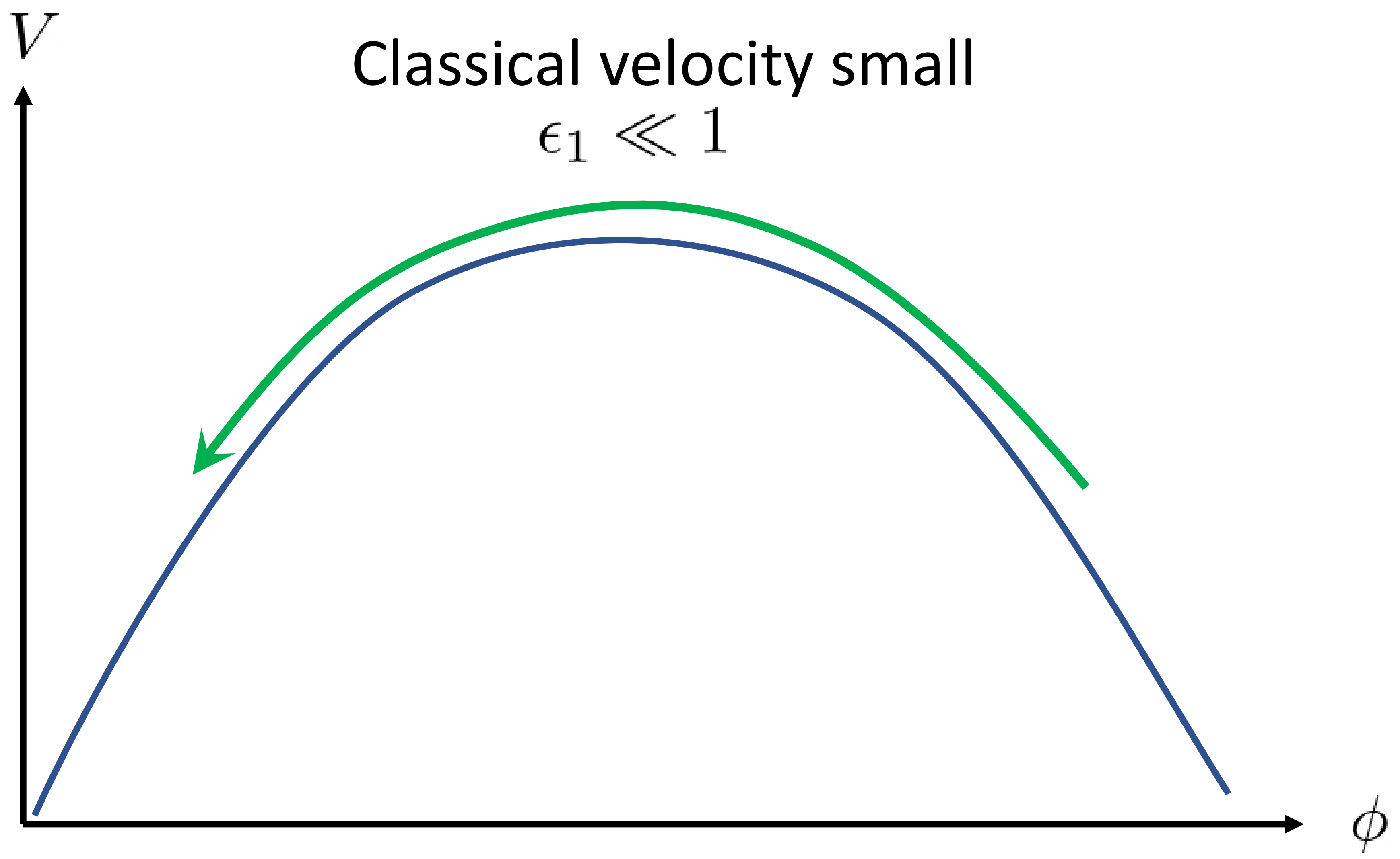
Full numerical computations

One million CPU hours



[2111.07437]

Zoom into the hilltop



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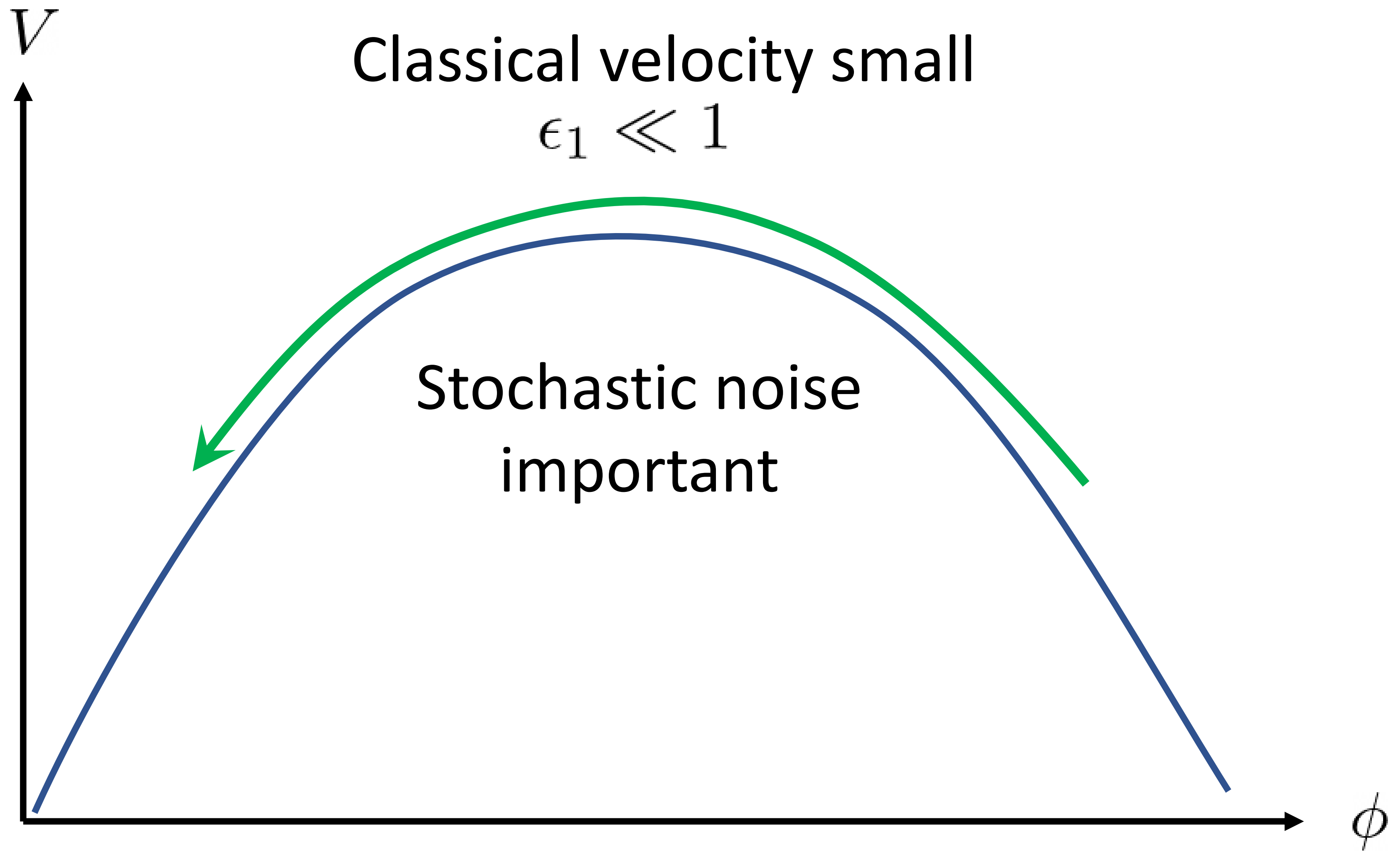

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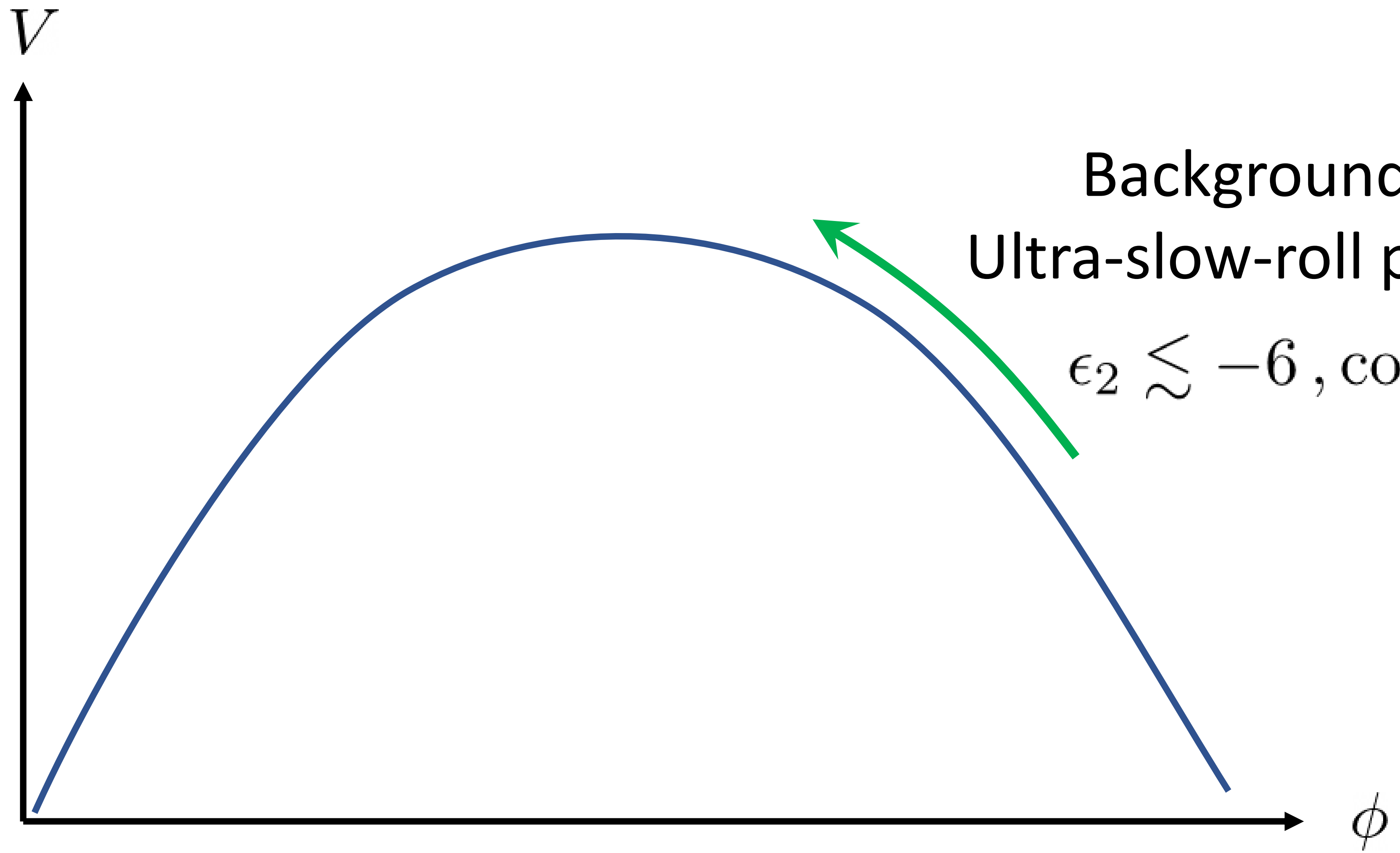
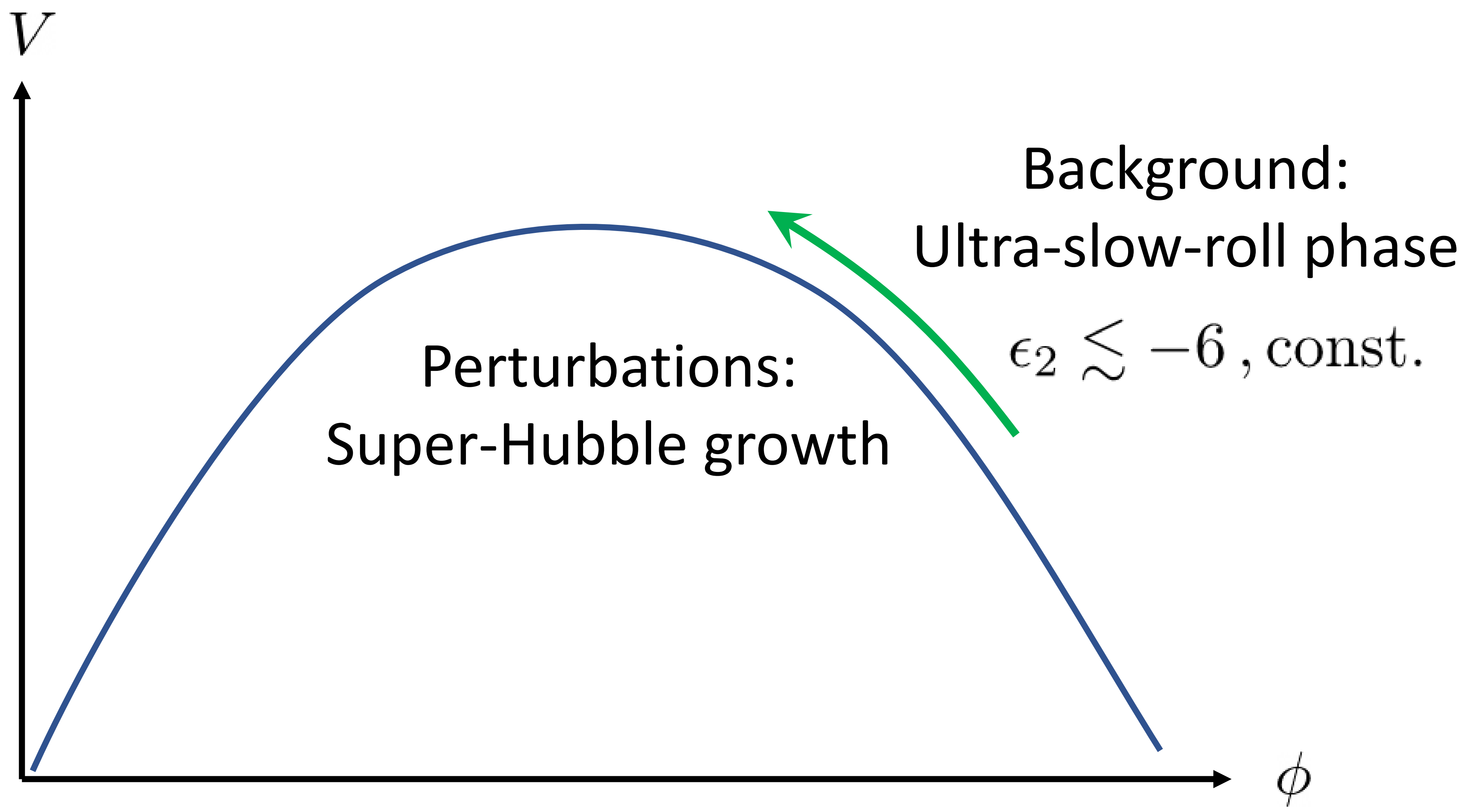


Background:
Ultra-slow-roll phase

$$\epsilon_2 \lesssim -6\,, \mathrm{const.}$$

Zoom into the hilltop

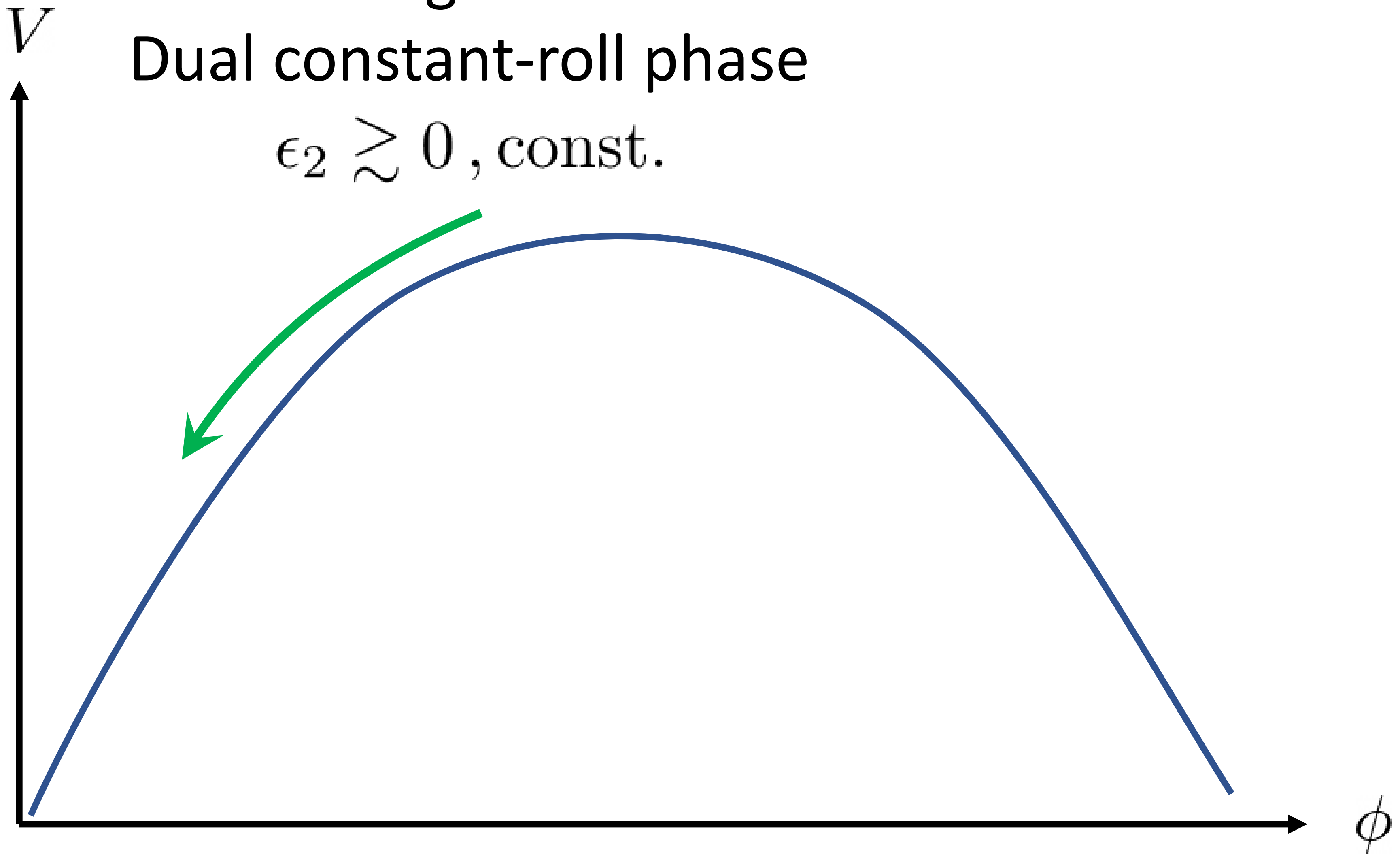


Zoom into the hilltop

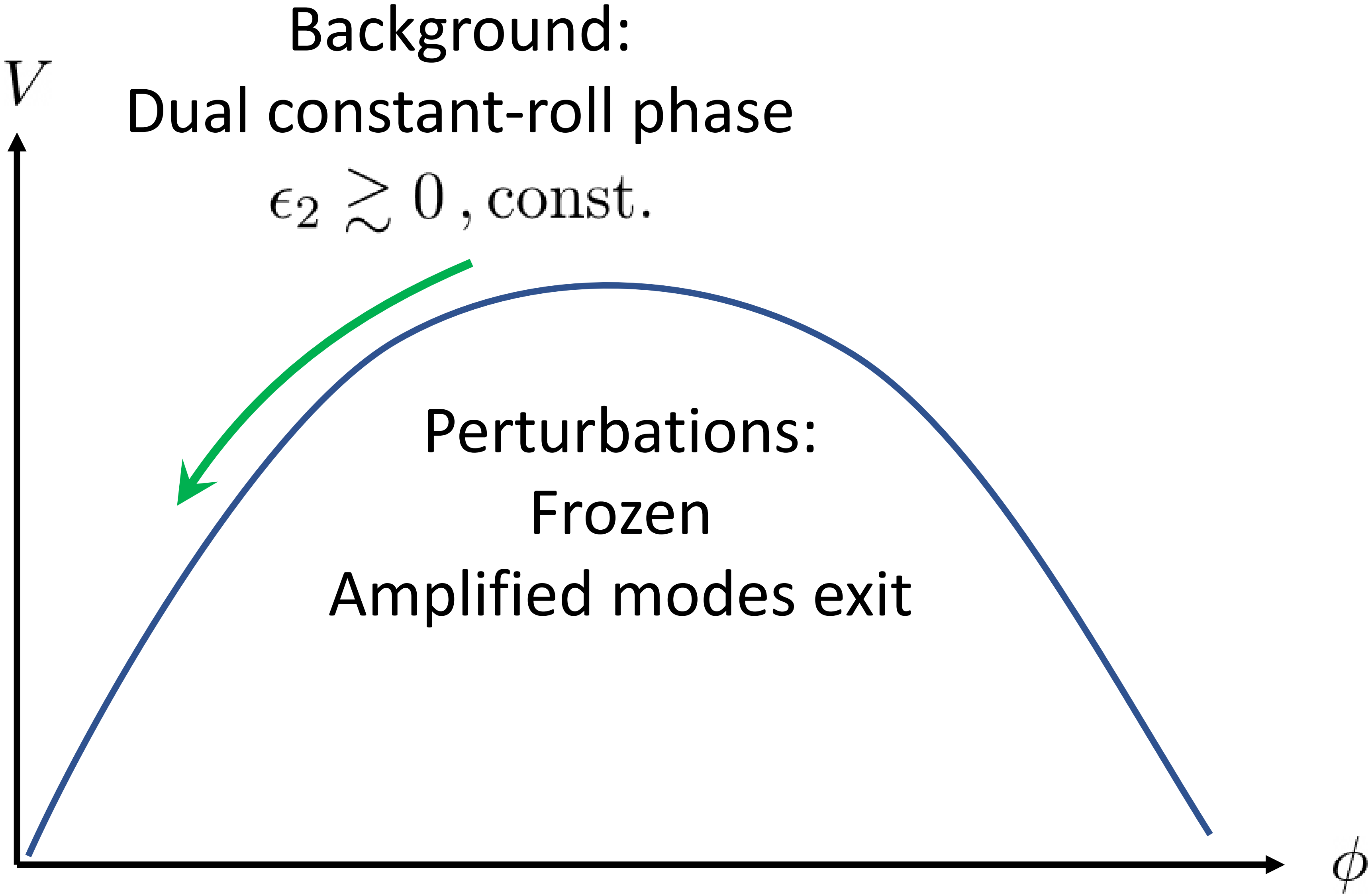
Background:

Dual constant-roll phase

$\epsilon_2 \gtrsim 0\,,\mathrm{const.}$



Zoom into the hilltop



Equations simplify in dual constant-roll phase

Adiabatic perturbations:
motion along classical trajectory only

Noise independent of background stochasticity:
pre-compute power spectrum

Simplified stochastic equation:

$$\mathrm{d}\phi = \frac{\epsilon_2}{2}(\phi - \phi_0)\mathrm{d}N + \frac{\epsilon_2}{2}\phi_0 e^{\frac{\epsilon_2}{2}N}\sqrt{\mathcal{P}_{\mathcal{R}}(k_\sigma)\mathrm{d}N}\,\hat{\xi}_N$$

$$\langle \hat{\xi}_N \hat{\xi}_{N'} \rangle = \delta_{NN'}$$

Simplified stochastic equation:

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$$\langle \hat{\xi}_N \hat{\xi}_{N'} \rangle = \delta_{NN'}$$

$$\phi(N) = \phi_0\left(1 - e^{\frac{\epsilon_2}{2}N}\right) + \frac{\epsilon_2}{2}\phi_0 e^{\frac{\epsilon_2}{2}N} X_{<k_\sigma}$$

$$X_{<k} \equiv \sum_{\tilde{k}=k_{\mathrm{ini}}}^{k} \sqrt{\mathcal{P}_\mathcal{R}(\tilde{k})\,\mathrm{d}\ln k}\,\hat{\xi}_{\tilde{k}}$$

ΔN distribution

$$p(X_{<k}) = \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{X_{<k}^2}{2\sigma_k^2}}, \quad \sigma_k^2 \equiv \int_{k_{\mathrm{ini}}}^{k} \mathcal{P}_{\mathcal{R}}(\tilde{k})\, \mathrm{d}\ln\tilde{k}$$

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$$X_{<k} = \frac{2}{\epsilon_2}\left(1 - e^{-\frac{\epsilon_2}{2}\Delta N_{<k}}\right)$$

ΔN distribution

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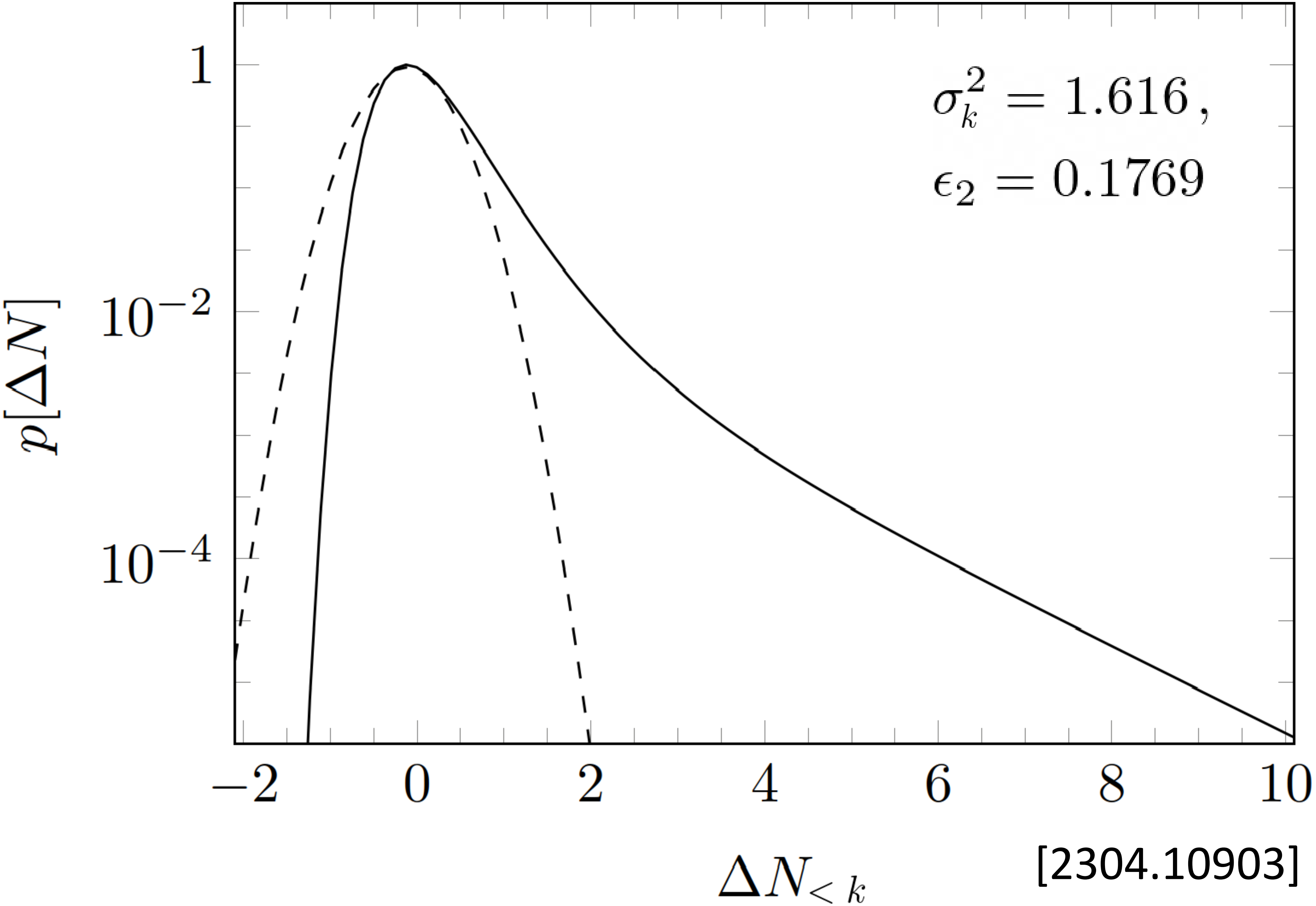
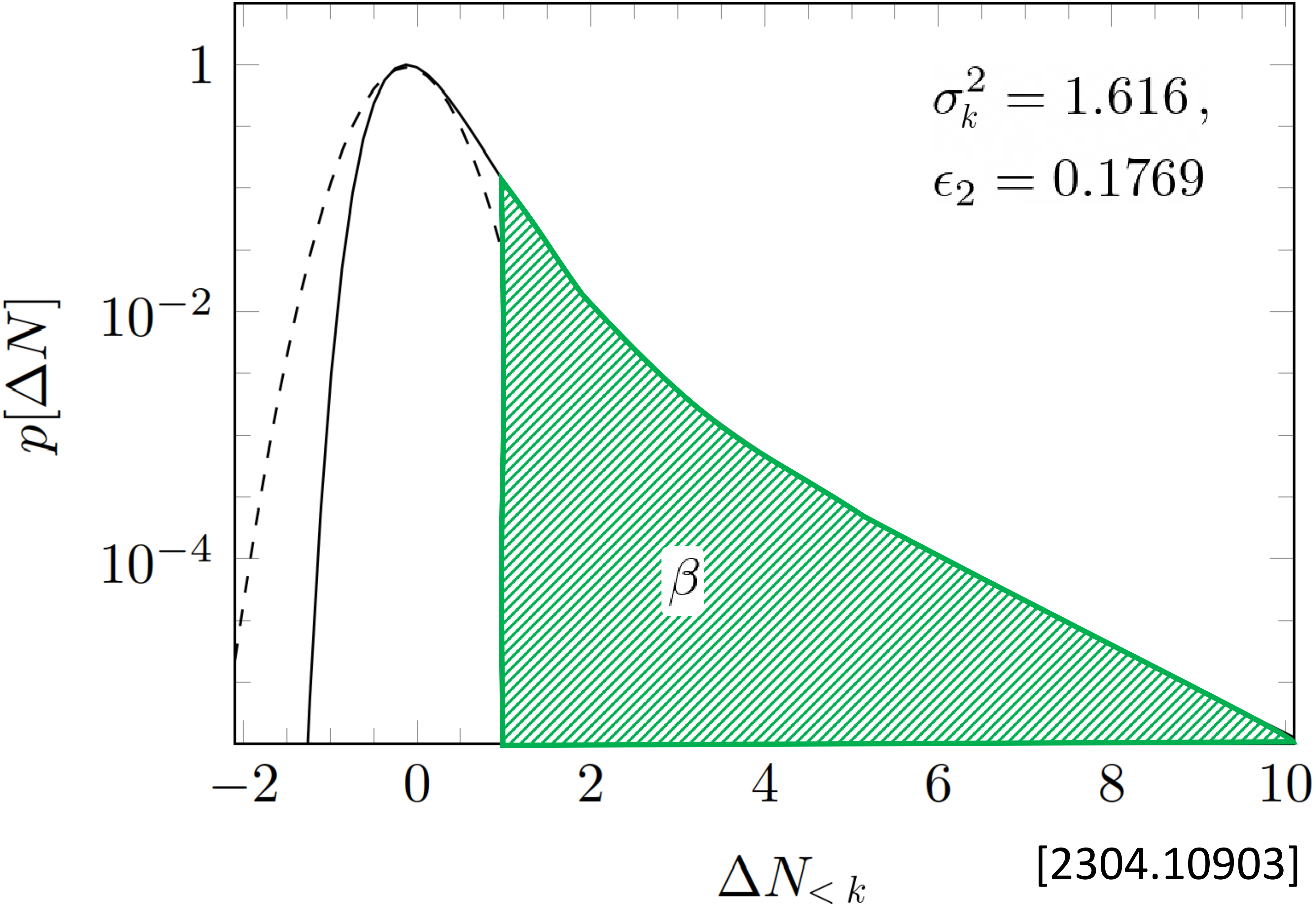
$$X_{<k} = \frac{2}{\epsilon_2}\left(1 - e^{-\frac{\epsilon_2}{2}\Delta N_{<k}}\right)$$

$$p(\Delta N_{<k}) = \frac{1}{\sqrt{2\pi}\sigma_k} \exp\left[-\frac{2}{\sigma_k^2\epsilon_2^2}\left(1 - e^{-\frac{\epsilon_2}{2}\Delta N_{<k}}\right)^2 - \frac{\epsilon_2}{2}\Delta N_{<k}\right]$$

$\Delta N_{<k} = \mathcal{R}_{<k}$


$\sigma_k^2 = 1.616\,,$
$\epsilon_2 = 0.1769$
$p[\Delta N]$
1
10^{-2}
10^{-4}
-2
0
2
4
6
8
10
$\Delta N_{<k}$

[2304.10903]



$\sigma_k^2 = 1.616\,,$
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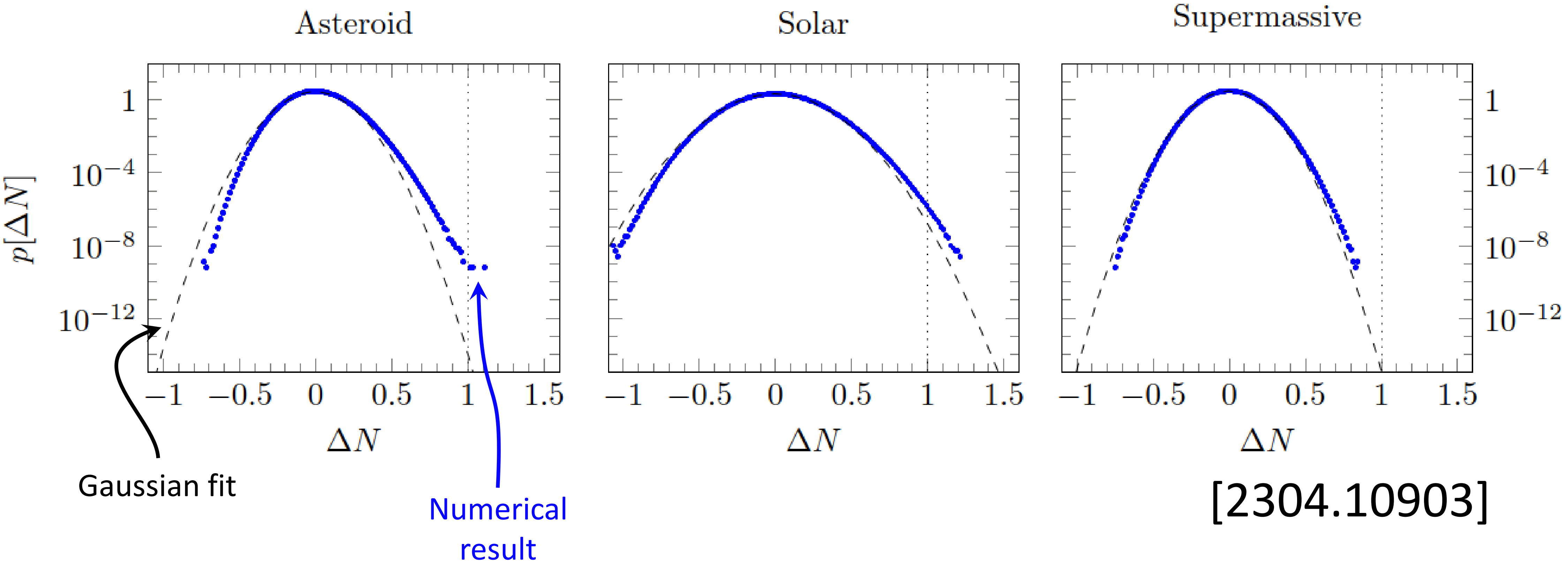
β
$p[\Delta N]$
1
10^{-2}
10^{-4}
−2
0
2
4
6
8
10
$\Delta N_{<\,k}$

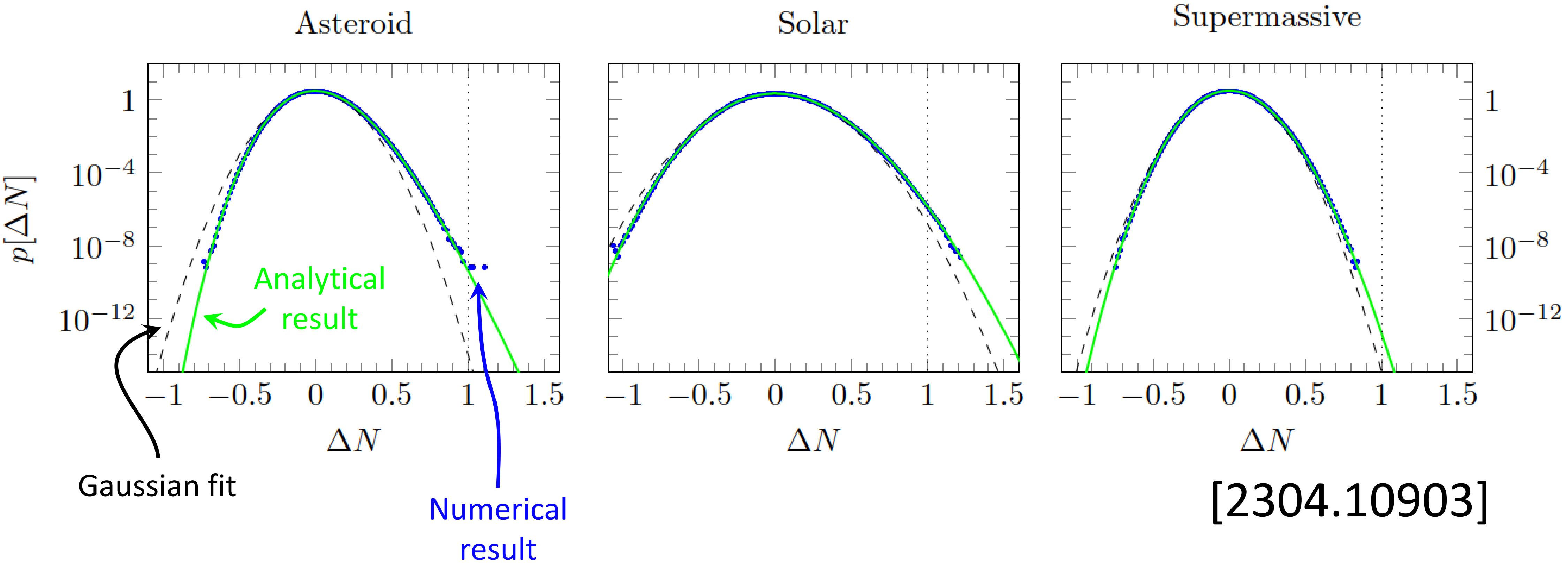
[2304.10903]

Comparison to numerics



[2304.10903]

Comparison to numerics



[2304.10903]

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Compaction function: right tool for determining the collapse threshold

$$\mathcal{C} \equiv 2\frac{M_{\rm MS} - M_{\rm bg}}{R}$$

Collapse: $\mathcal{C}_{\rm max} > \mathcal{C}_c \approx 0.4$

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$$\mathcal{C} \equiv 2\frac{M_{\rm MS} - M_{\rm bg}}{R}$$

Collapse: $\mathcal{C}_{\rm max} > \mathcal{C}_c \approx 0.4$

In inflationary variables:

$$\mathcal{C}(r) = \frac{2}{3}(1 - [1 + r\zeta'(r)]^2)$$

Assume spherical symmetry

$$r\zeta'(r) = \sum_k \frac{2k^2\,\mathrm{d}k}{\sqrt{2\pi}}\,\zeta_k\left[\cos(kr) - \frac{\sin(kr)}{kr}\right]$$

$$\zeta_k = \frac{\sqrt{2\pi}}{2k^3}\,\frac{\mathrm{d}\zeta_{<k}}{\mathrm{d}\ln k}$$

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$$\zeta_k = \frac{\sqrt{2\pi}}{2k^3}\frac{\mathrm{d}\zeta_{<k}}{\mathrm{d}\ln k}$$

Vary k:
Full profile
in one patch of space!

Recall: in the stochastic picture,

$$\zeta_{<k} = \Delta N_{<k} = -\frac{2}{\epsilon_2}\ln\left(1 - \frac{\epsilon_2}{2}X_{<k}\right) = -\frac{2}{\epsilon_2}\ln\left(1 - \frac{\epsilon_2}{2}\sum_{\tilde{k}=k_{\mathrm{ini}}}^{k}\sqrt{\mathcal{P}_{\mathcal{R}}(\tilde{k})\,\mathrm{d}\ln k}\,\hat{\xi}_{\tilde{k}}\right)$$

Master formula

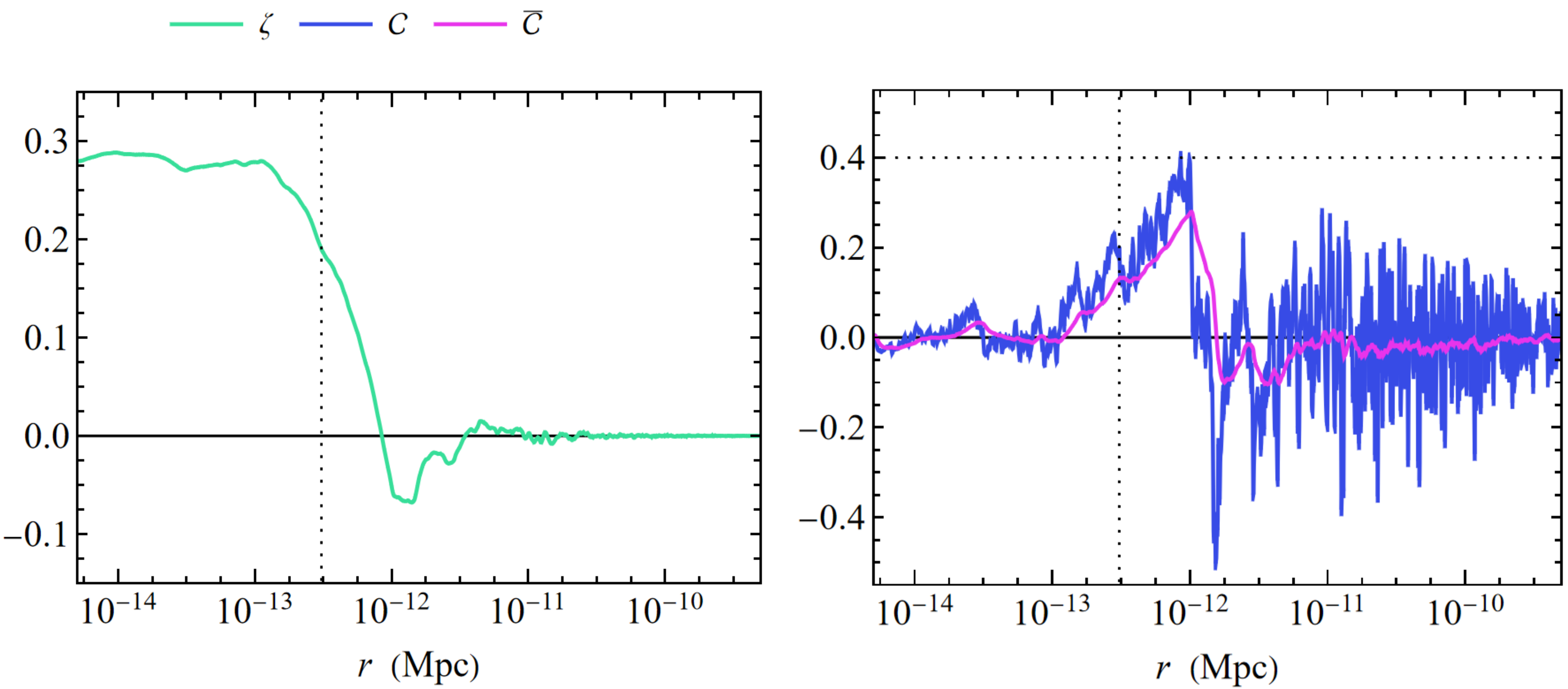
$$r\zeta'(r) = \sum_k \left[-\frac{\hat{\xi}_k}{1-\frac{\epsilon_2}{2}X_{<k}}\sqrt{\mathcal{P}_\zeta(k)\,\mathrm{d}\ln k} + \frac{\epsilon_2}{4\left(1-\frac{\epsilon_2}{2}X_{<k}\right)^2}\mathcal{P}_\zeta(k)\,\mathrm{d}\ln k \right] \times \left[\cos(kr) - \frac{\sin(kr)}{kr}\right]$$

Alternative collapse measure: averaged compaction function

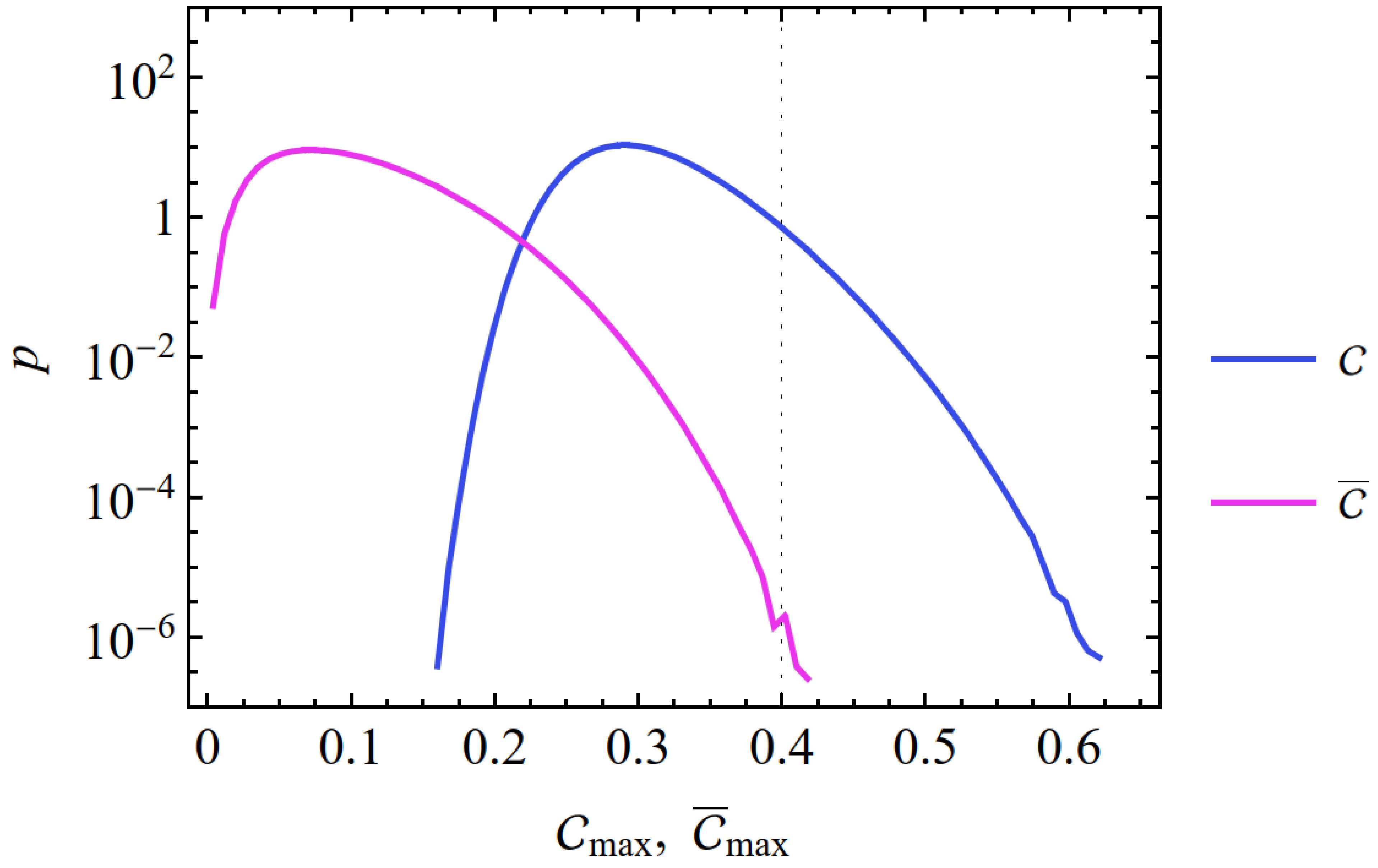
$$R = are^{\zeta}$$

$$\bar{\mathcal{C}}(r) \equiv \frac{3}{R(r)^3} \int_0^{R(r)} \mathrm{d}\tilde{R}\tilde{R}^2\mathcal{C}$$

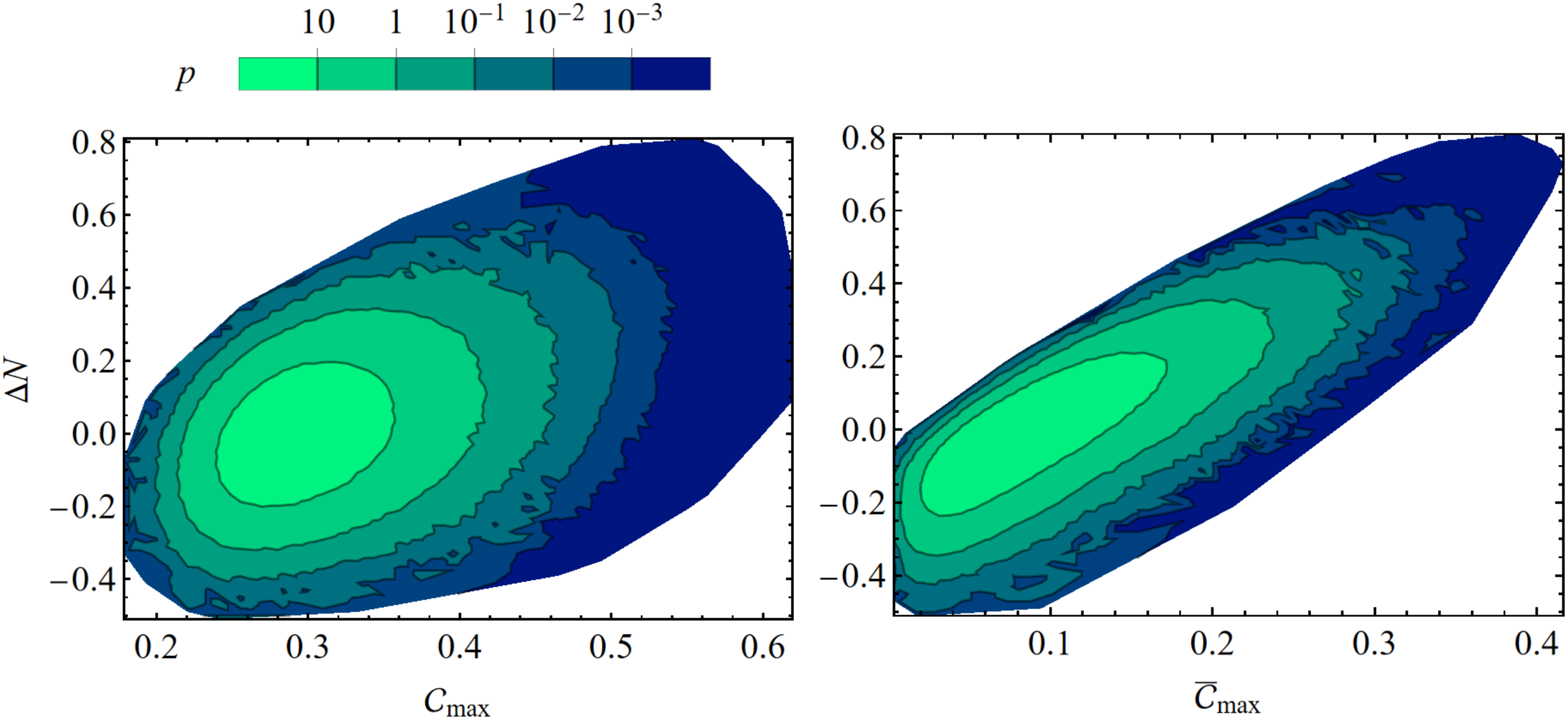
$$= -\frac{2}{r^3 e^{3\zeta(r)}} \int_0^{r} \mathrm{d}\tilde{r}\, \tilde{r}^2 e^{3\zeta} [2\tilde{r}\zeta' + 3(\tilde{r}\zeta')^2 + (\tilde{r}\zeta')^3]$$



ζ
C
C̄
0.3
0.2
0.1
0.0
−0.1
0.4
0.2
0.0
−0.2
−0.4
10^{-14}
10^{-13}
10^{-12}
10^{-11}
10^{-10}
r (Mpc)
r (Mpc)



10^2
1
10^{-2}
10^{-4}
10^{-6}
p
0
0.1
0.2
0.3
0.4
0.5
0.6
$C_{\max}$, $\overline{C}_{\max}$
C
$\overline{C}$



p
10
1
10^{-1}
10^{-2}
10^{-3}
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
ΔN
0.2
0.3
0.4
0.5
0.6
$C_{\max}$
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
0.1
0.2
0.3
0.4
$\overline{C}_{\max}$

Initial PBH fractions

Gaussian approximation, $\mathcal{R}_{<k} > 1$, fixed k: $\beta \approx 5 \times 10^{-16}$

Non-Gaussian statistics, $\mathcal{R}_{<k} > 1$, fixed k: $\beta \approx 2.2 \times 10^{-11}$

$\bar{\mathcal{C}}_{\max} > 0.4: \quad \beta \approx 1.4 \times 10^{-8}$

$\mathcal{C}_{\max} > 0.4: \quad \beta \approx 0.016$

Problems

Collapse simulations have smooth peaks.

Us: Stochastic peaks?

- Numeric converge: ok
- Physics? Smoothing? Window functions?

Multiple peaks?

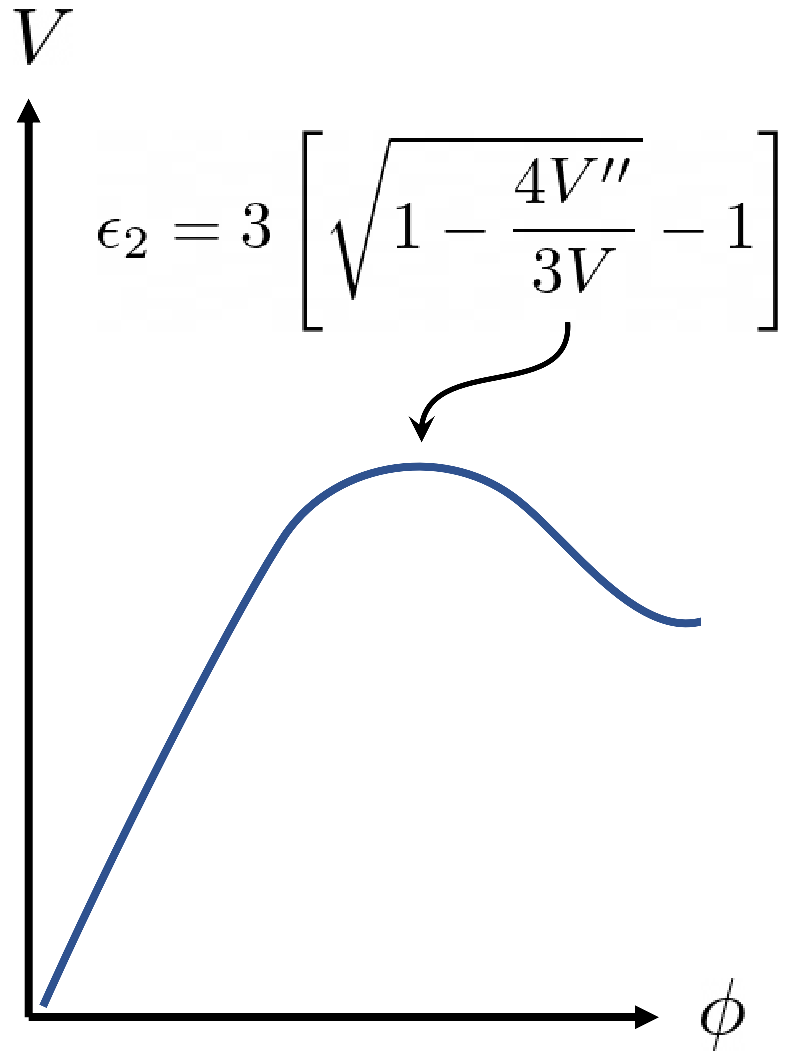
- “Outermost peak” gives final collapse?
- Overlapping peaks?

Conclusions

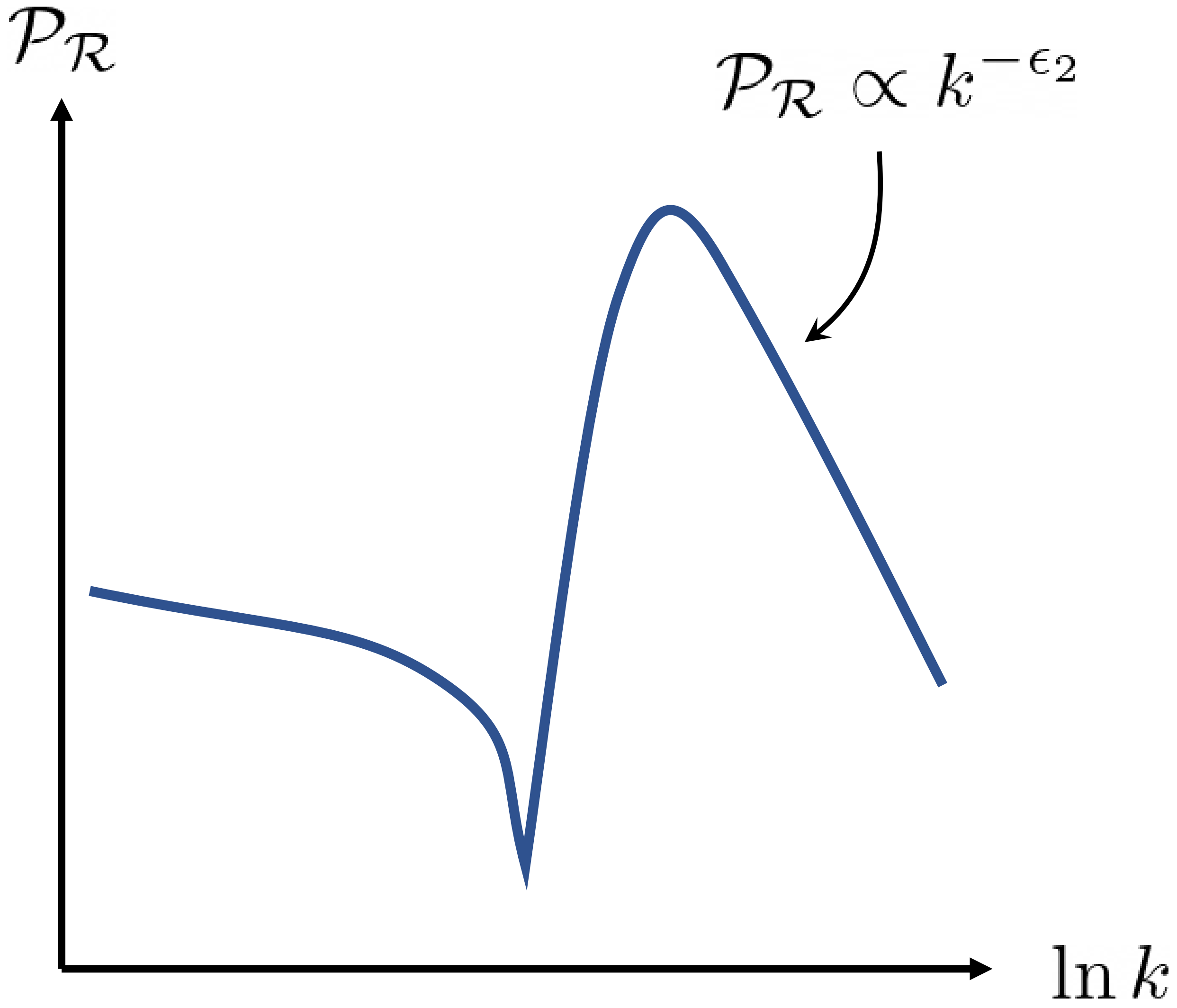
Stochastic inflation introduces non-Gaussian corrections to PBH statistics

Compaction function formalism needed for accurate results

Spiked radial profiles: what to do?

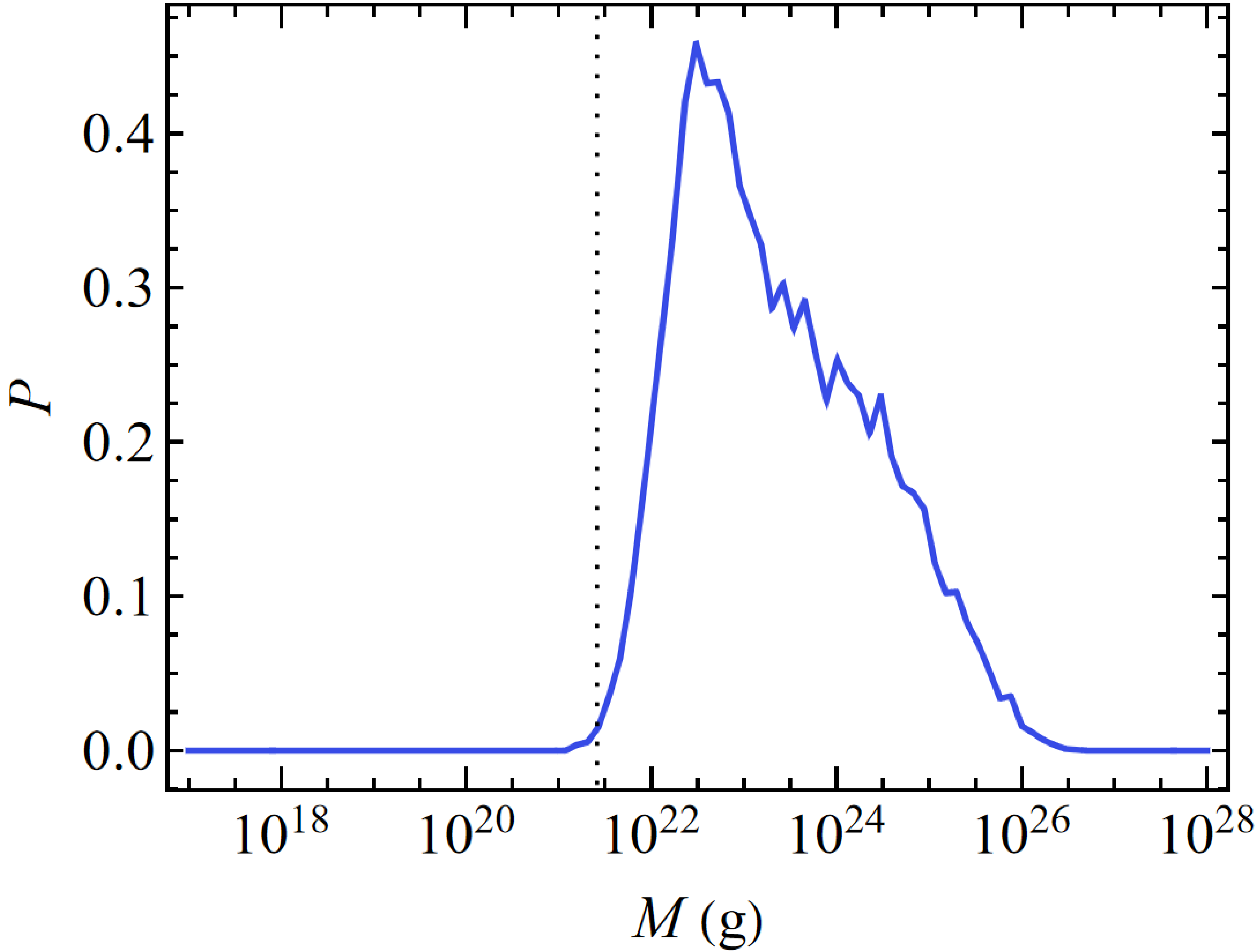


V
$\epsilon_2 = 3\left[\sqrt{1-\frac{4V''}{3V}}-1\right]$
ϕ



$\mathcal{P}_\mathcal{R}$
$\mathcal{P}_\mathcal{R} \propto k^{-\epsilon_2}$
$\ln k$

[2205.13540]



0.4
0.3
0.2
0.1
0.0
P
10^18
10^20
10^22
10^24
10^26
10^28
M (g)