Stochastic constant-roll inflation and primordial black holes

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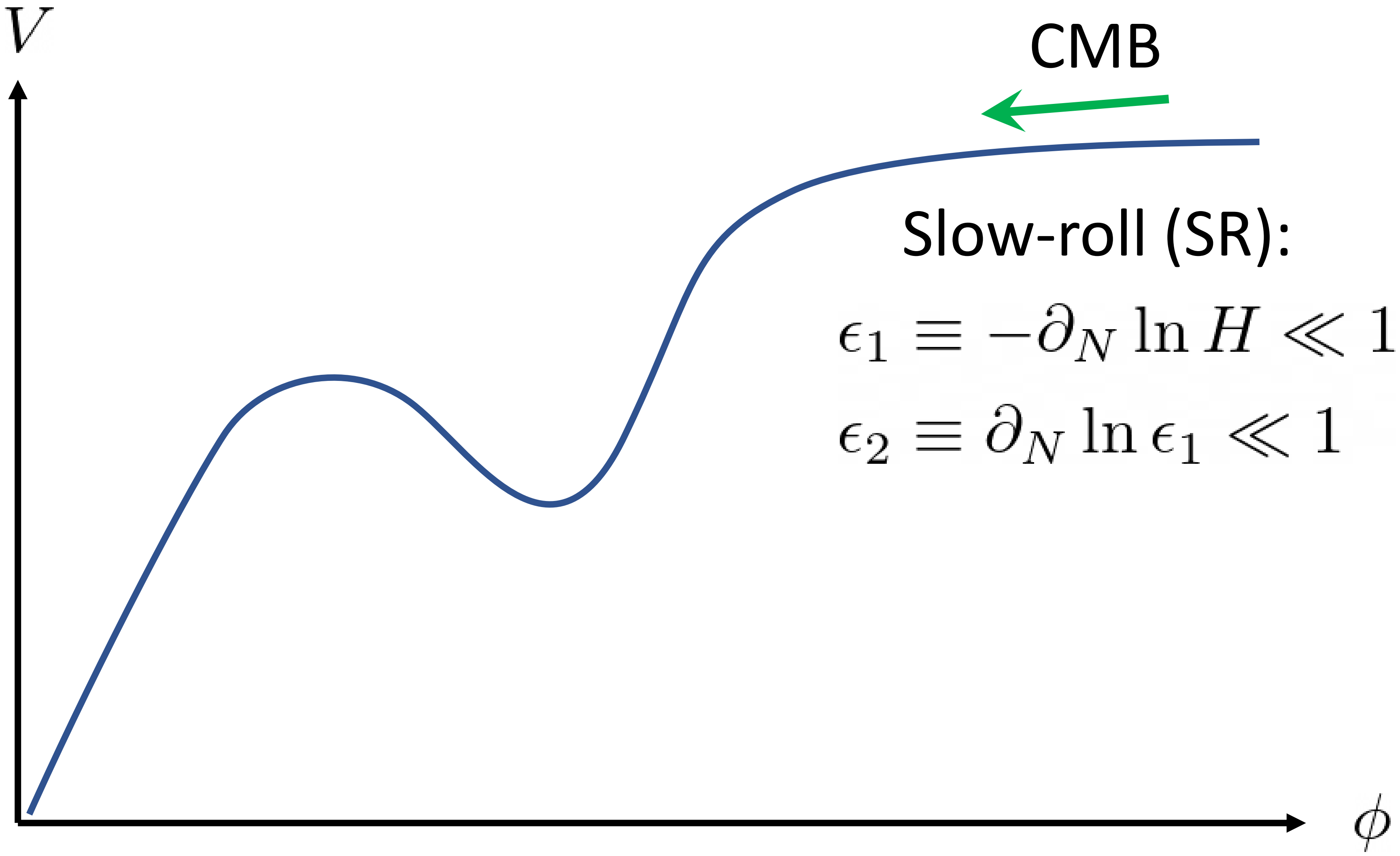
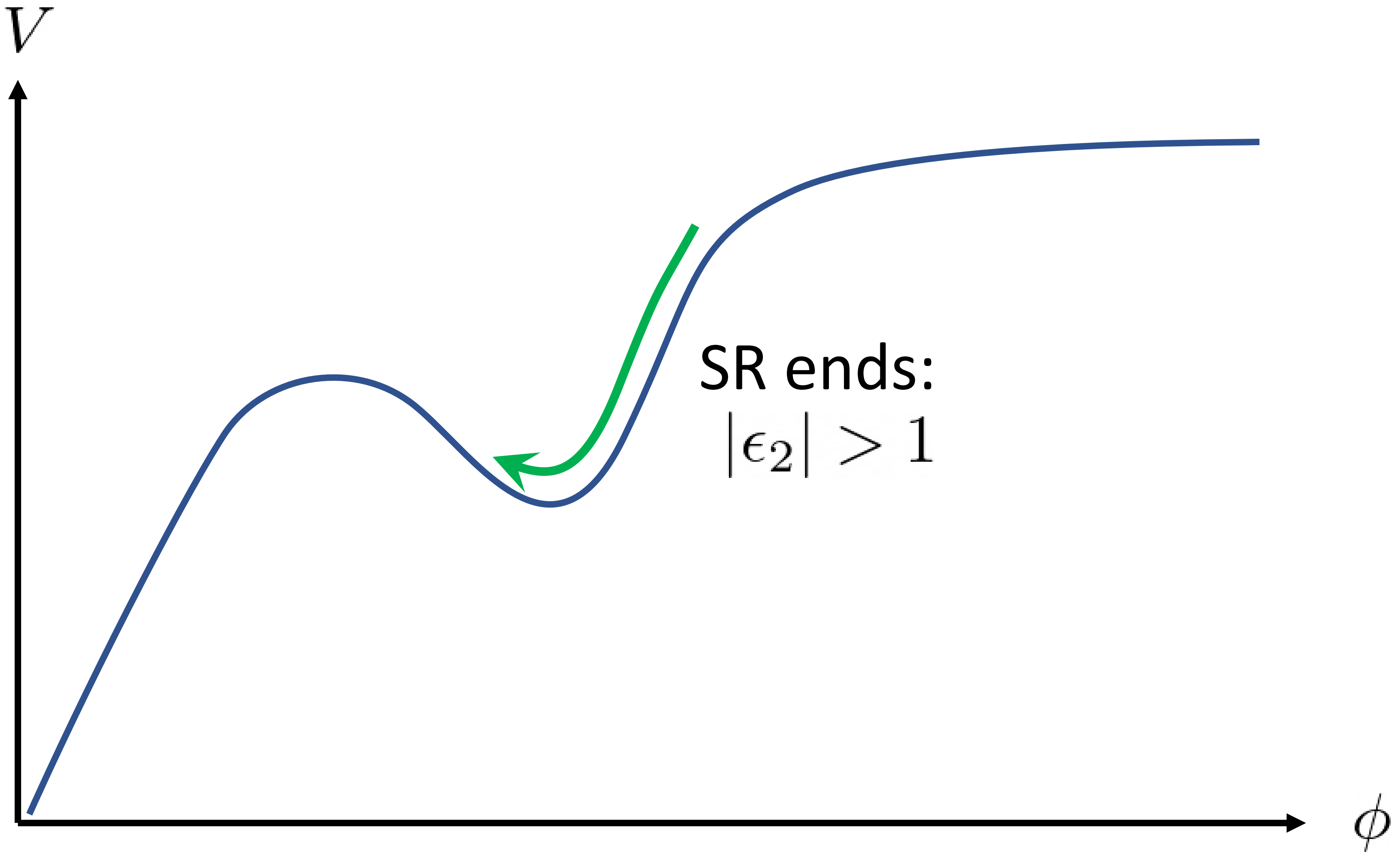
Eemeli Tomberg, NICPB Tallinn

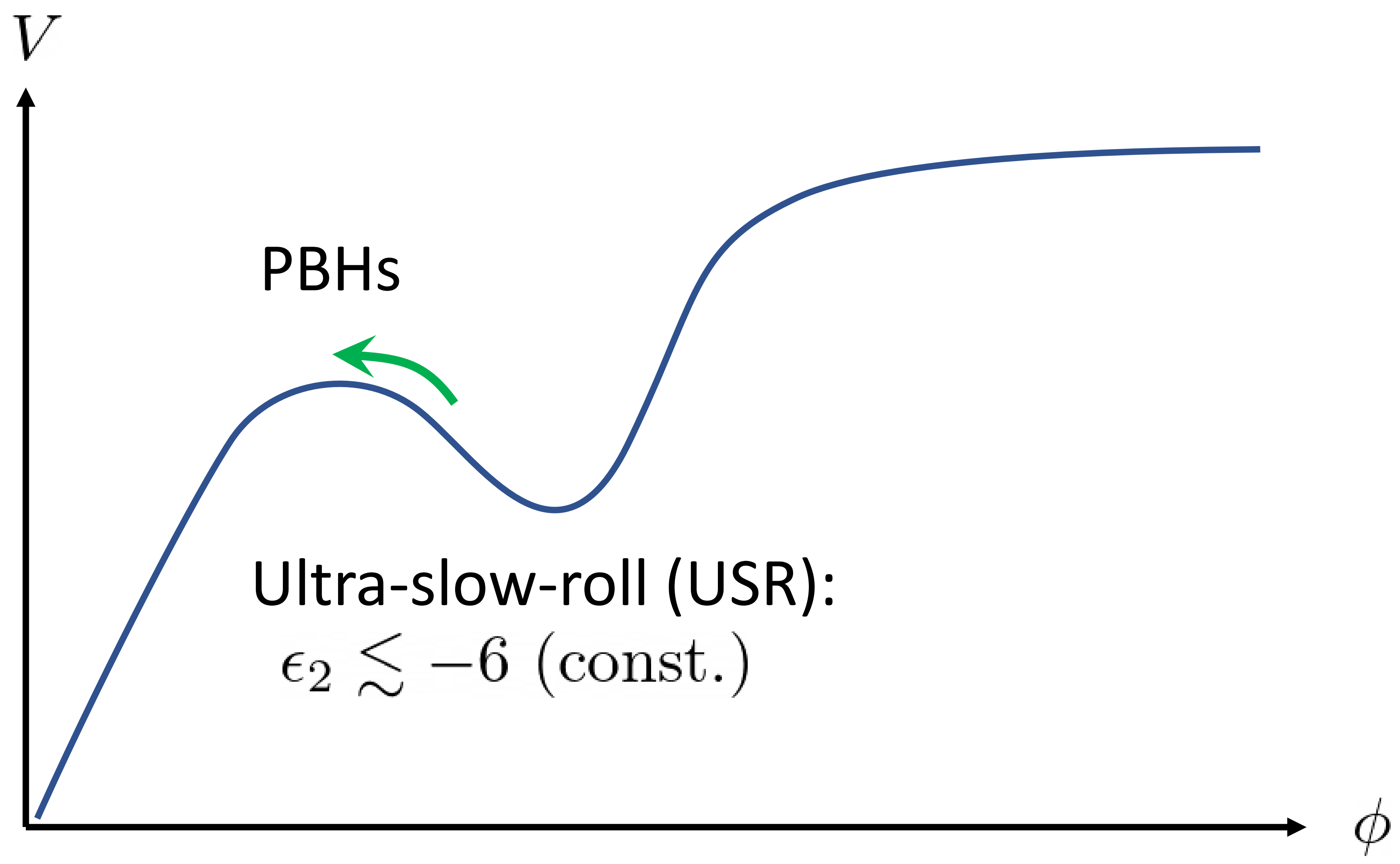
Based on 2012.06551, 2111.07437, 2210.17441, 2304.10903
in collaboration with D. Figueroa, S. Raatikainen, S. Räsänen



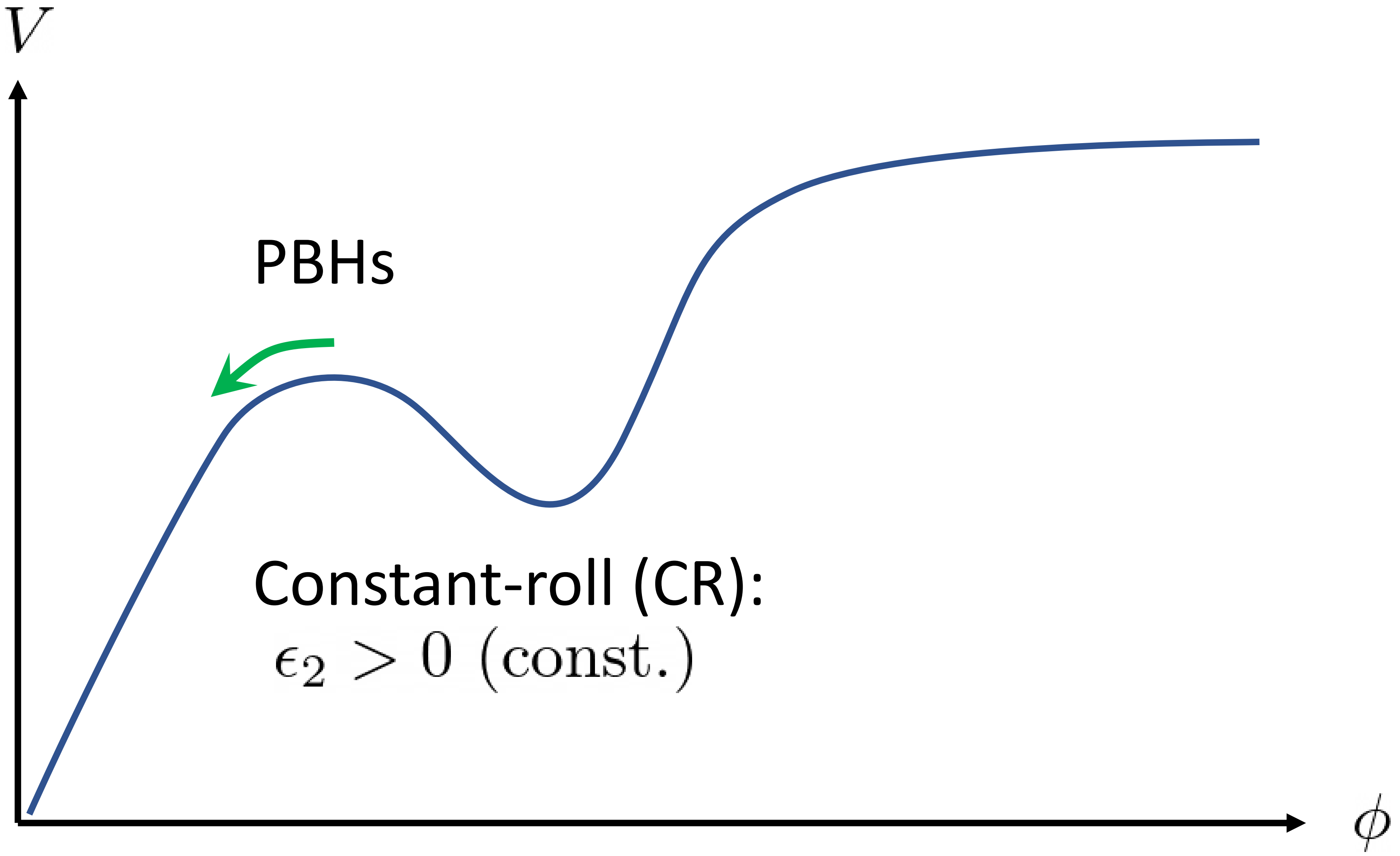
V
CMB
Slow-roll (SR):
$\epsilon_1 \equiv -\partial_N \ln H \ll 1$
$\epsilon_2 \equiv \partial_N \ln \epsilon_1 \ll 1$
ϕ



V
SR ends:
$|\epsilon_2| > 1$
ϕ

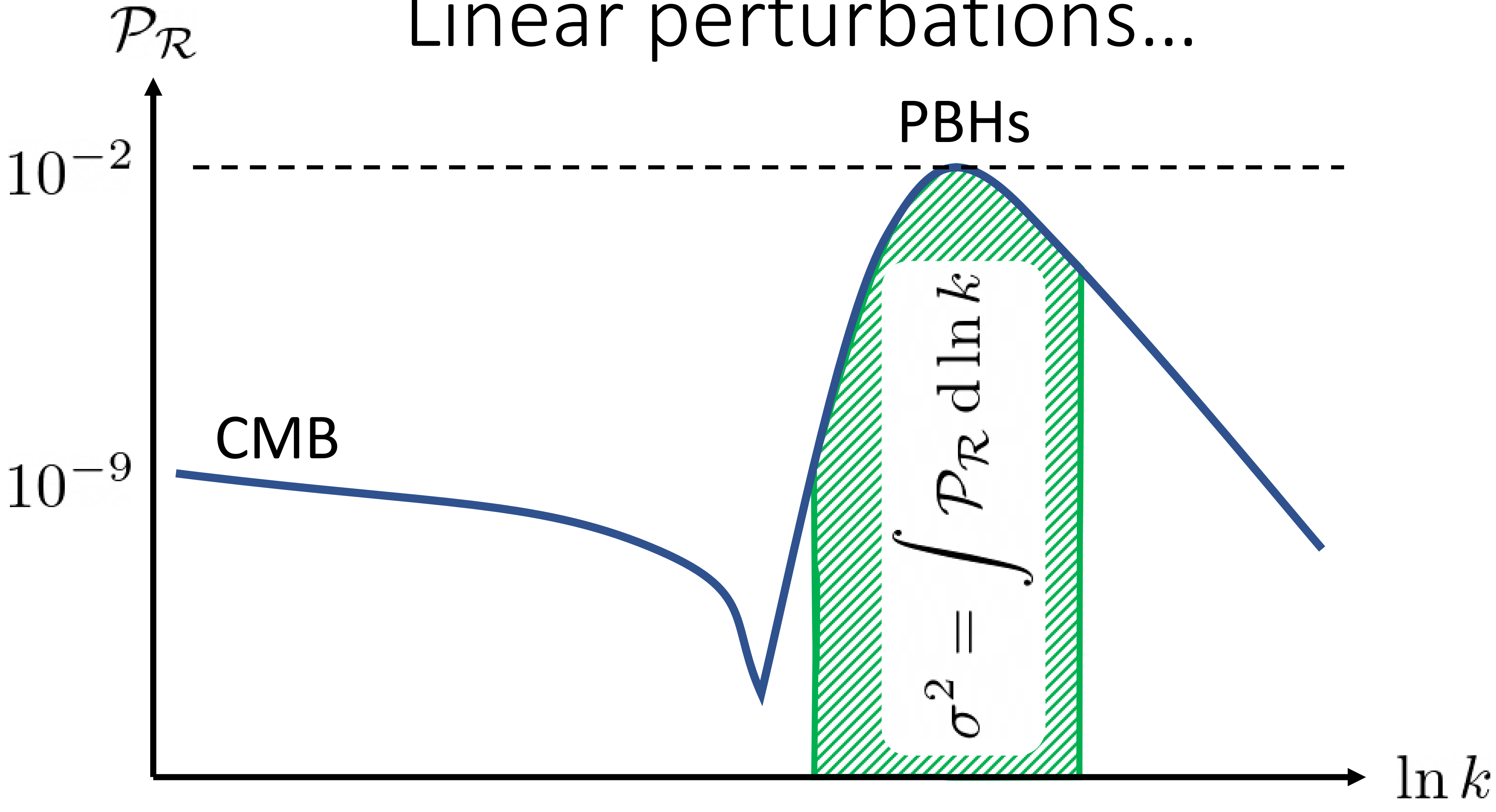


V
PBHs
Ultra-slow-roll (USR):
$\epsilon_2 \lesssim -6$ (const.)
ϕ

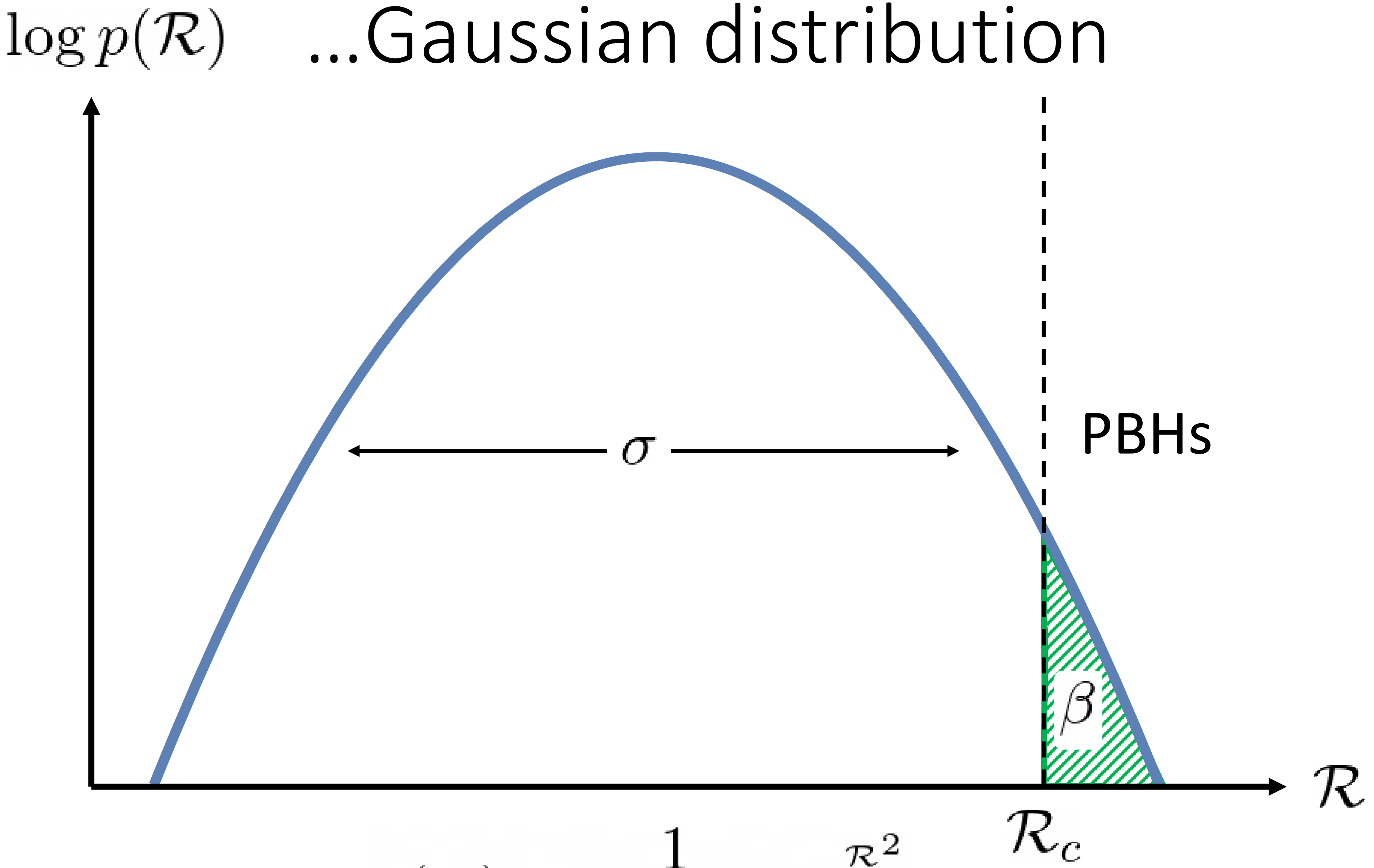


V
PBHs
Constant-roll (CR):
$\epsilon_2 > 0$ (const.)
ϕ

Linear perturbations…



…Gaussian distribution



$$p(\mathcal{R}) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\mathcal{R}^2}{2\sigma^2}}$$

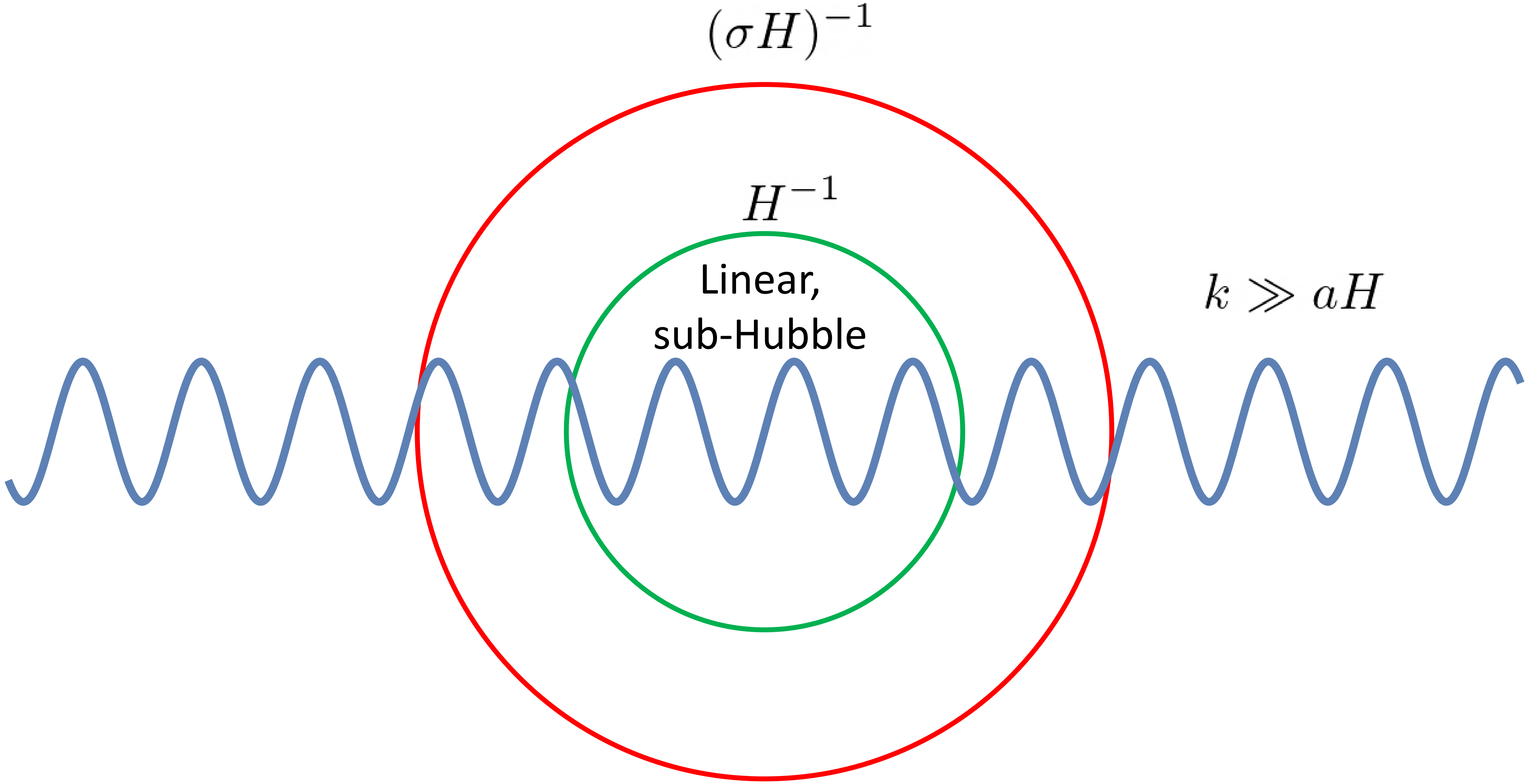
Stochastic inflation

Inflaton field: $\phi + \delta\phi$

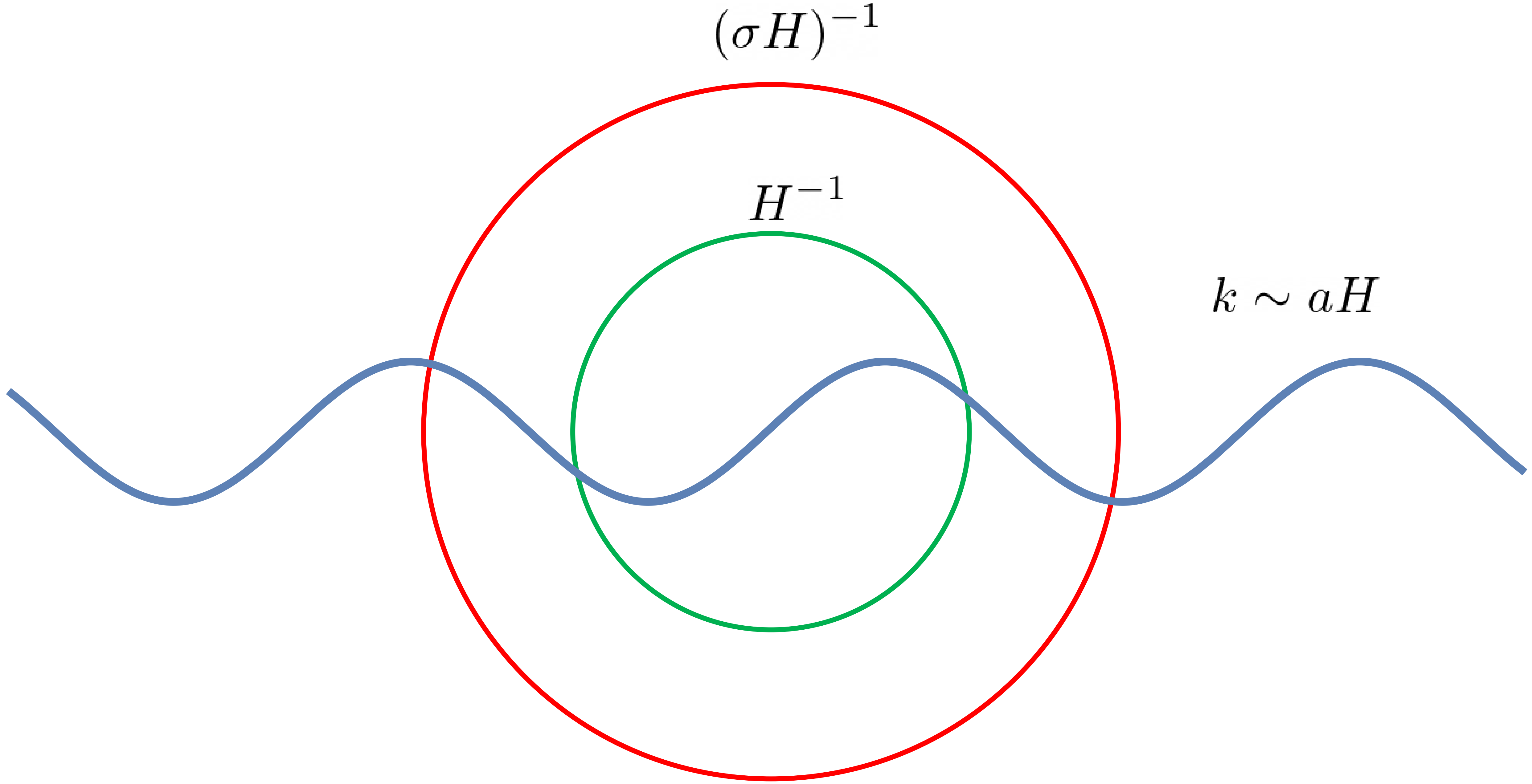
Coarse-grained:
FLRW

Short-wavelength:
linear perturbation theory

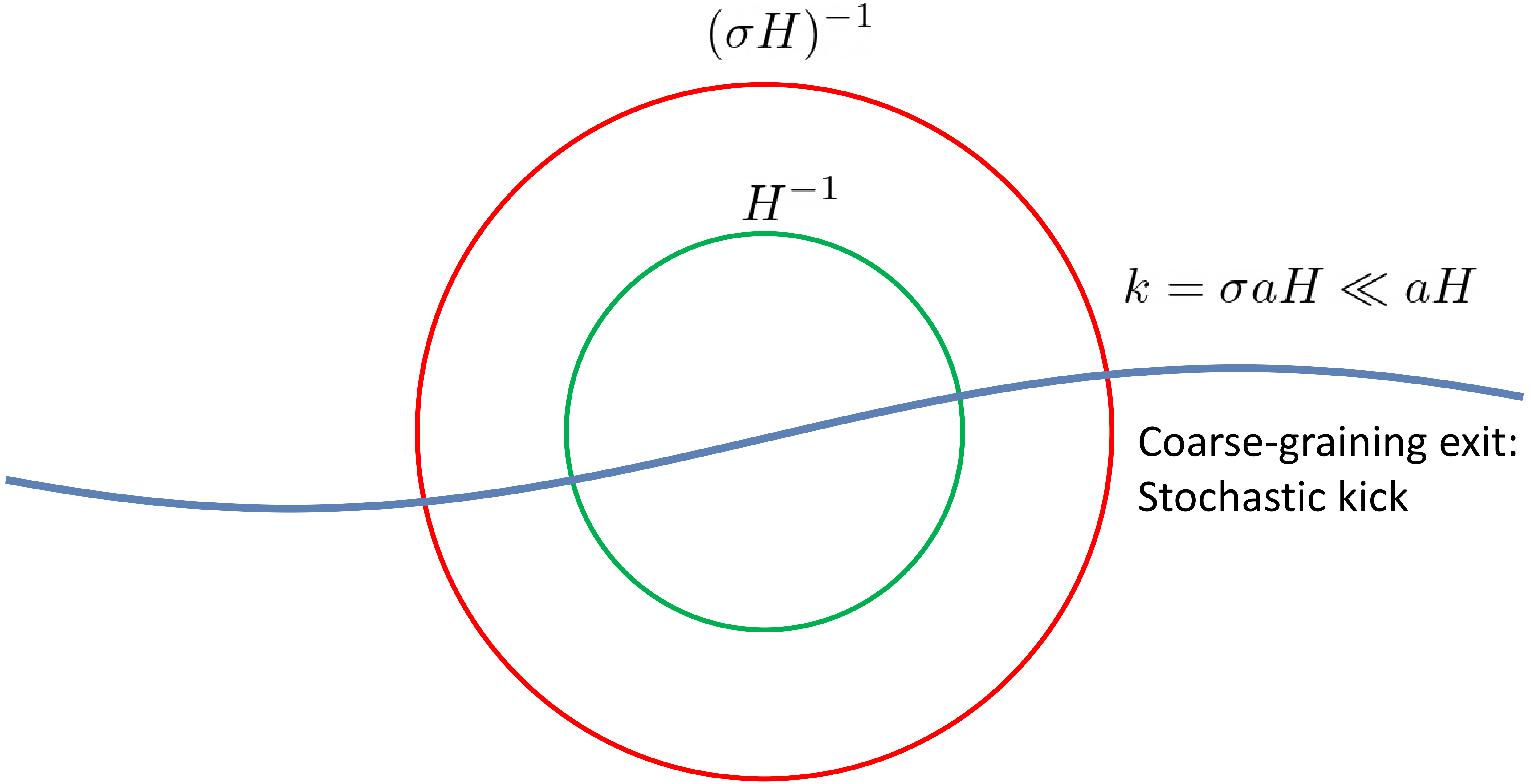
Patched together at the coarse-graining scale $k = k_\sigma \equiv \sigma a H$



$(\sigma H)^{-1}$
H^{-1}
Linear,
sub-Hubble
$k \gg aH$



$(\sigma H)^{-1}$
H^{-1}
$k \sim aH$


$(\sigma H)^{-1}$
H^{-1}
$k = \sigma a H \ll aH$
Coarse-graining exit:
Stochastic kick

Stochastic inflation

$$\phi' = \pi + \xi_\phi \,, \quad \pi' = -\left(3 - \frac{1}{2}\pi^2\right)\pi - \frac{V'(\phi)}{H^2} + \xi_\pi \,, \quad H^2 = \frac{V(\phi)}{3 - \frac{1}{2}\pi^2}$$

$$\delta\phi_k'' = -(3 - \frac{1}{2}\pi^2)\delta\phi_k' - \left[\frac{k^2}{a^2H^2} + \pi^2(3 - \frac{1}{2}\pi^2) + 2\pi\frac{V'(\phi)}{H^2} + \frac{V''(\phi)}{H^2}\right]\delta\phi_k$$

$$\langle\xi_\phi(N)\xi_\phi(N')\rangle = \frac{1}{6\pi^2}\frac{\mathrm{d}k_\sigma^3}{\mathrm{d}N}|\delta\phi_{k_\sigma}(N)|^2\delta(N-N')$$

$$\langle\xi_\pi(N)\xi_\pi(N')\rangle = \frac{1}{6\pi^2}\frac{\mathrm{d}k_\sigma^3}{\mathrm{d}N}|\delta\phi'_{k_\sigma}(N)|^2\delta(N-N')$$

$$\langle\xi_\phi(N)\xi_\pi(N')\rangle = \frac{1}{6\pi^2}\frac{\mathrm{d}k_\sigma^3}{\mathrm{d}N}\delta\phi_{k_\sigma}(N)\delta\phi'^*_{k_\sigma}(N)\delta(N-N')$$

$$\mathcal{R} = \Delta N \equiv N - \bar{N}$$

Numerical method

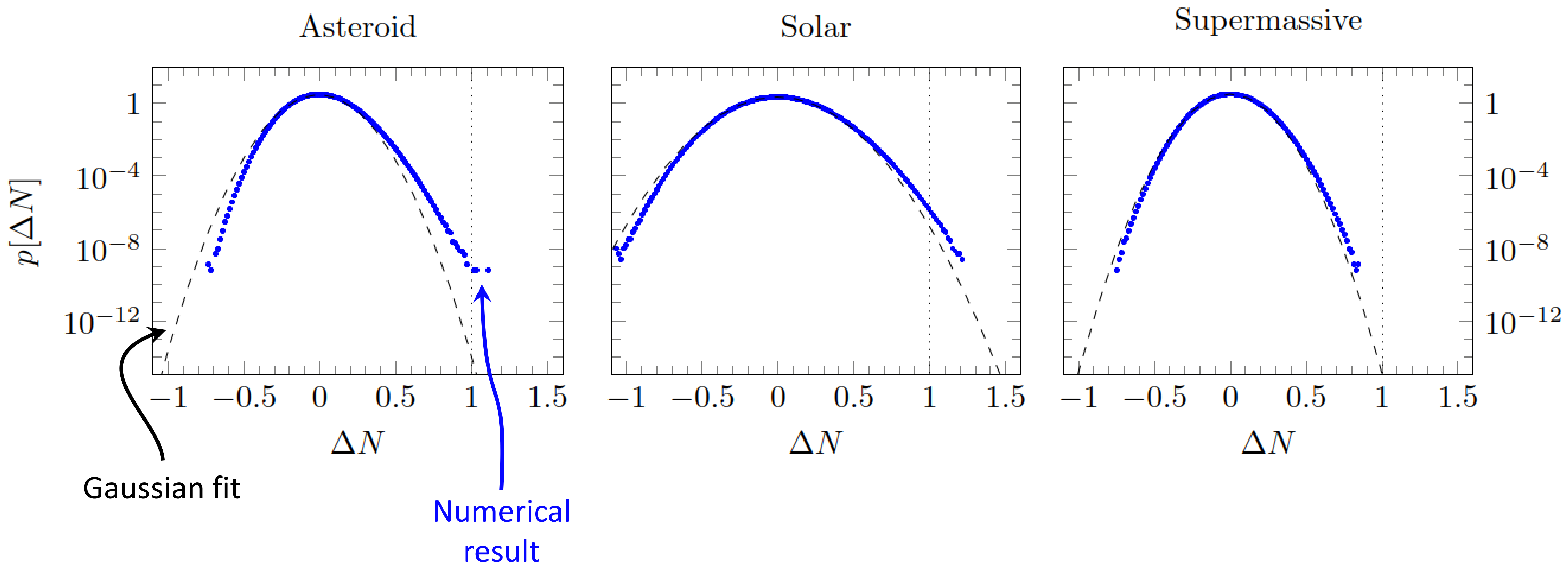
[Figueroa et al, 2012.06551]
[Figueroa et al, 2111.07437]

Noise: beyond de Sitter approximation, $|\delta\phi_{k_\sigma}|^2 \neq \frac{H^2}{2k_\sigma^3}$

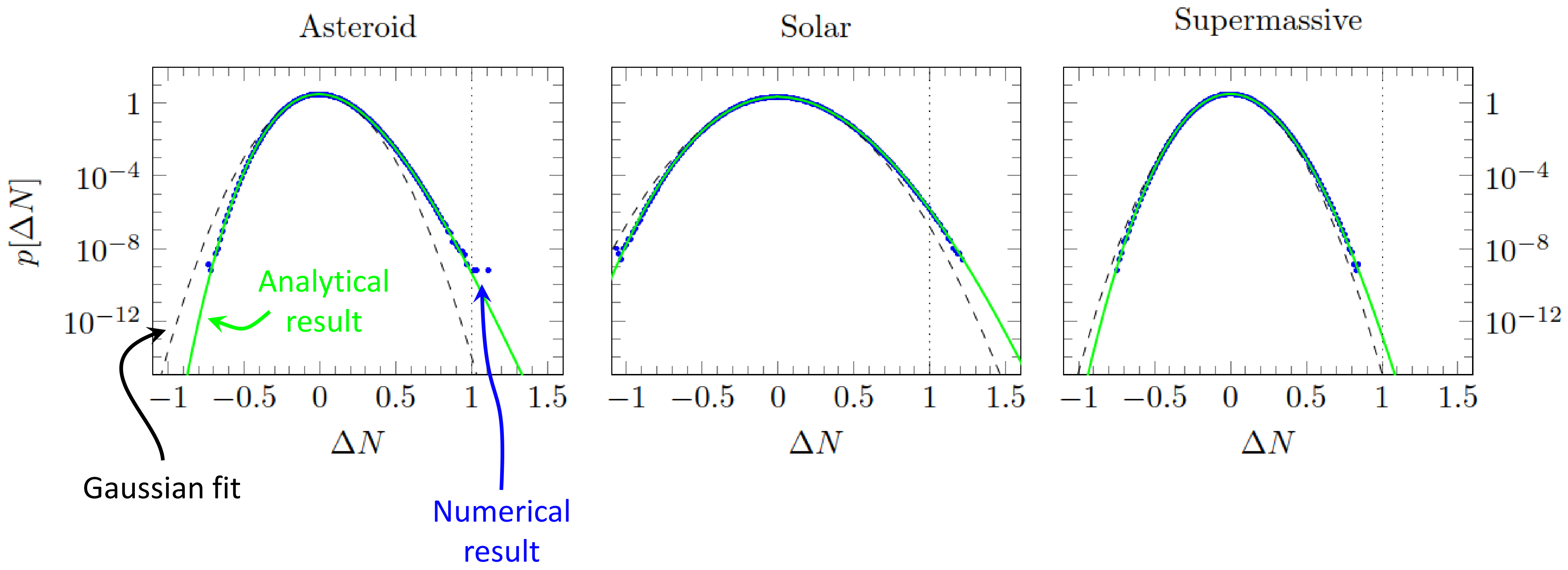
Turn stochastic kicks off at fixed N, giving the PBH scale (slightly different from FPT)

Collect statistics: a million CPU hours

[Tomberg, 2304.10903]



[Tomberg, 2304.10903]



Stochastic constant-roll inflation [Tomberg, 2304.10903]

Power spectrum peak modes:
Hubble exit ($k = aH$) during USR ($\epsilon_2 < -6\,, \mathrm{const.}$)
Coarse-graining ($k = \sigma a H$) later, in CR ($\epsilon_2 > 0\,, \mathrm{const.}$)

Allows simplifications

Motion constrained to one dimension

Curvature perturbations squeezed:

$$\xi_\pi = \xi_\phi \frac{\delta\phi'_k}{\delta\phi_k}$$

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Curvature perturbations frozen:

$$\mathcal{R}_k = \frac{\delta\phi_k}{\sqrt{2\epsilon_1}} = \text{const.} \qquad \Rightarrow \qquad \frac{\delta\phi_k'}{\delta\phi_k} = \frac{\pi'}{\phi'} = \frac{1}{2}\epsilon_2$$

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System moves on classical background trajectory (like in SR):

$$\frac{\xi_\phi}{\xi_\pi} = \frac{\phi'}{\pi'}$$

Simple perturbation evolution

Perturbation evolution independent of stochasticity:

$$\delta\phi_k' = \frac{\epsilon_2}{2}\delta\phi_k$$

Pre-compute perturbations:

$$\frac{k^3}{2\pi^3}|\delta\phi_k(N)|^2 = \epsilon_1(N)\mathcal{P}_{\mathcal{R}}(k)$$

Simple classical evolution

$$\phi = \frac{2}{\epsilon_2}\pi + \phi_0 = (1 - e^{\frac{\epsilon_2}{2}N})\phi_0\,, \qquad \epsilon_1 = \frac{\epsilon_2^2}{4}\phi_0^2 e^{\epsilon_2 N}$$

Field linear in drift

Simplified stochastic equation:

$$\mathrm{d}\phi = \pi \mathrm{d}N + \xi_\phi \mathrm{d}N$$

Simplified stochastic equation:

$$\mathrm{d}\phi = \frac{\epsilon_2}{2}(\phi - \phi_0)\mathrm{d}N + \frac{\epsilon_2}{2}\phi_0 e^{\frac{\epsilon_2}{2}N}\sqrt{\mathcal{P}_{\mathcal{R}}(k_\sigma)\mathrm{d}N}\,\hat{\xi}_N$$

$$\langle \hat{\xi}_N \hat{\xi}_{N'} \rangle = \delta_{NN'}$$

Simplified stochastic equation:

$$\mathrm{d}\phi = \frac{\epsilon_2}{2}(\phi - \phi_0)\mathrm{d}N + \frac{\epsilon_2}{2}\phi_0 e^{\frac{\epsilon_2}{2}N}\sqrt{\mathcal{P}_{\mathcal{R}}(k_\sigma)\mathrm{d}N}\,\hat{\xi}_N$$

$$\langle \hat{\xi}_N \hat{\xi}_{N'} \rangle = \delta_{NN'}$$

$$\phi(N) = \phi_0\left(1 - e^{\frac{\epsilon_2}{2}N}\right) + \frac{\epsilon_2}{2}\phi_0 e^{\frac{\epsilon_2}{2}N} X(N)$$

$$X(N) \equiv \sum_{k=k_{\mathrm{ini}}}^{k=k_\sigma(N)} \sqrt{\mathcal{P}_{\mathcal{R}}(k)\,\mathrm{d}\ln k}\,\hat{\xi}_k$$

ΔN distribution

$$p(X) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{X^2}{2\sigma^2}} \,, \qquad \sigma^2 \equiv \int_{k_{\mathrm{ini}}}^{k_{\mathrm{end}}} \mathcal{P}_{\mathcal{R}}(k) \, \mathrm{d} \ln k$$

$$X = \frac{2}{\epsilon_2} \left(1 - e^{-\frac{\epsilon_2}{2} \Delta N} \right)$$

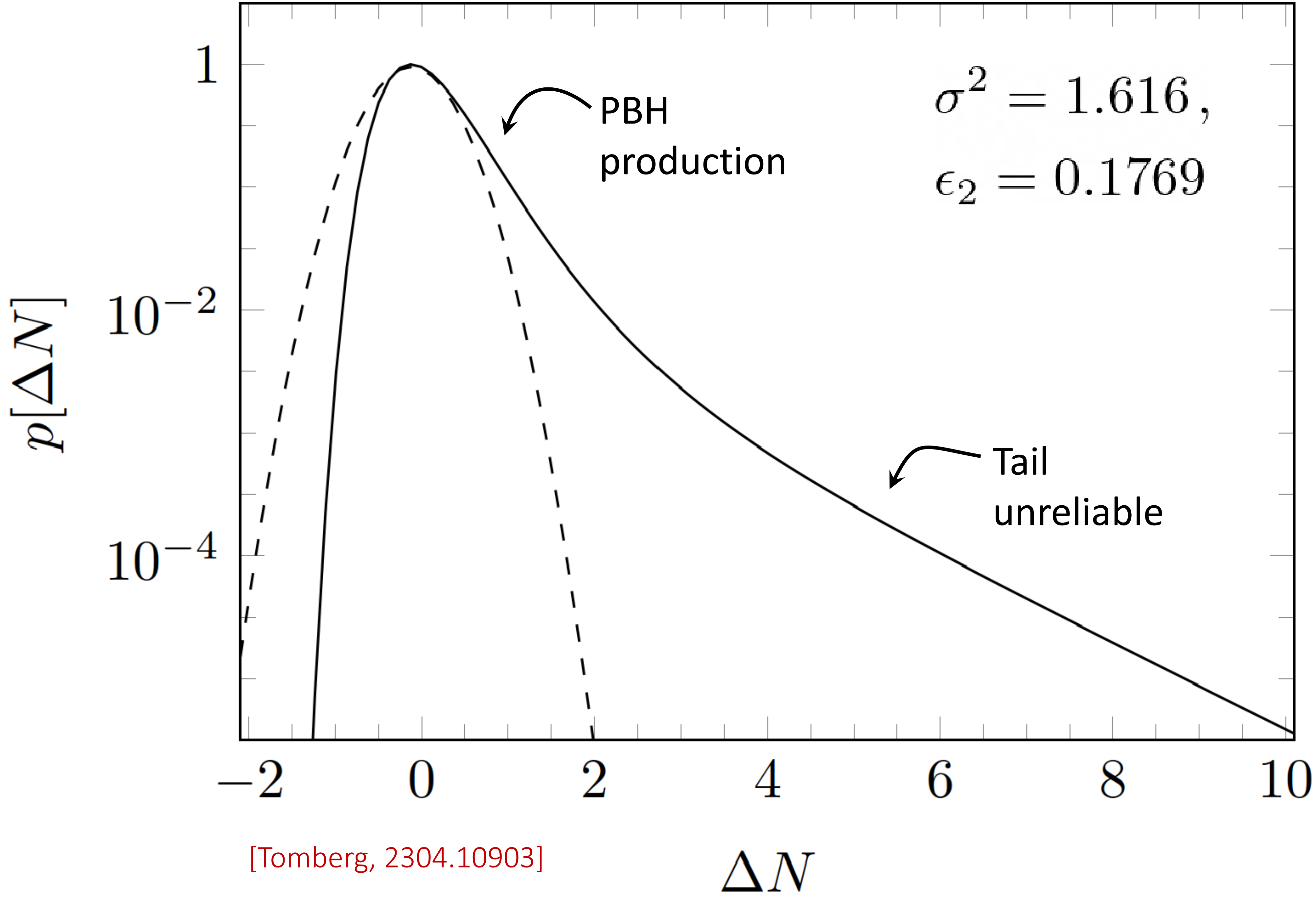
ΔN distribution

$$p(X) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{X^2}{2\sigma^2}}, \quad \sigma^2 \equiv \int_{k_{\rm ini}}^{k_{\rm end}} \mathcal{P}_{\mathcal{R}}(k)\, \mathrm{d}\ln k$$

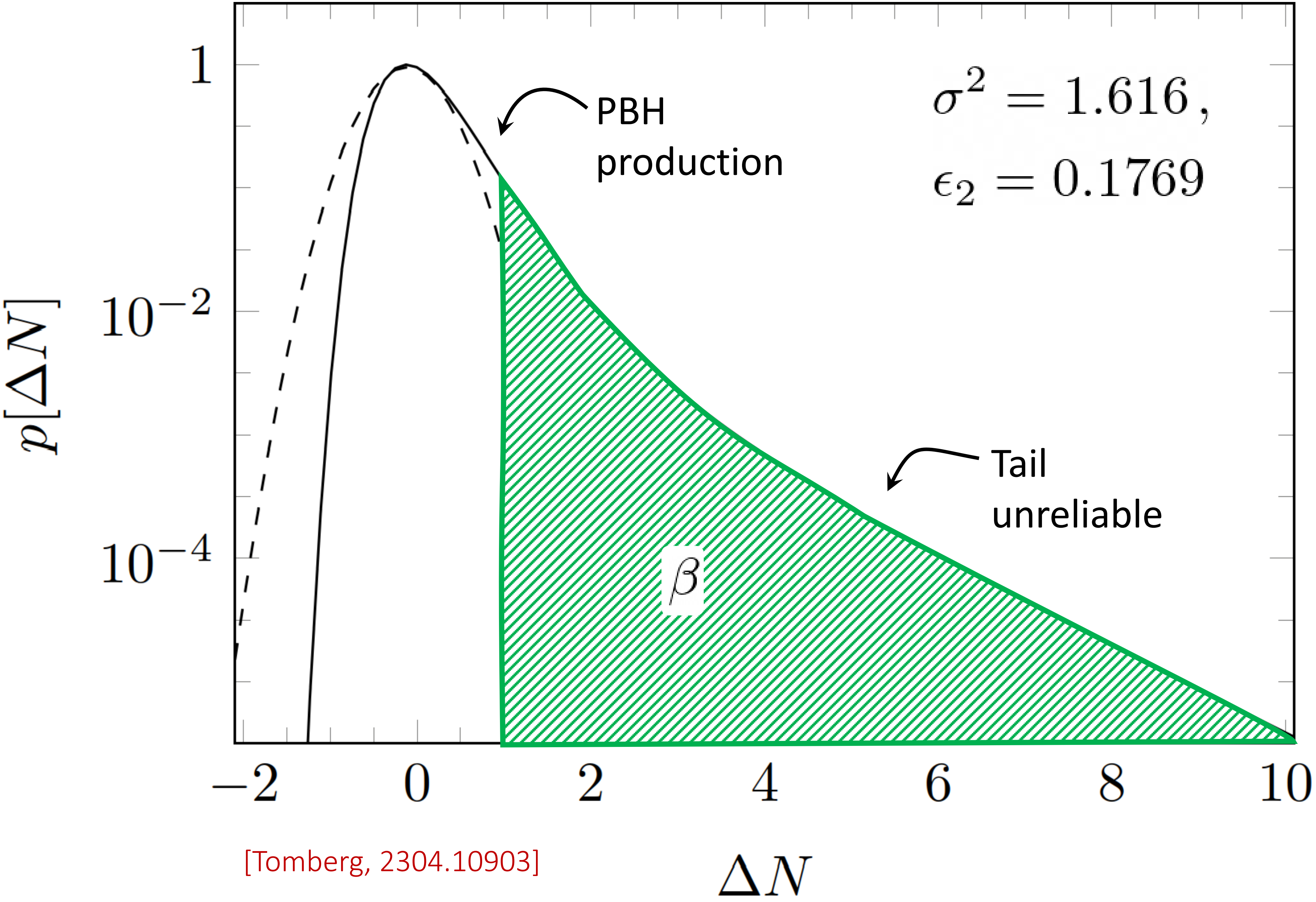
$$X = \frac{2}{\epsilon_2}\left(1 - e^{-\frac{\epsilon_2}{2}\Delta N}\right)$$

$$p(\Delta N) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{2}{\sigma^2\epsilon_2^2}\left(1 - e^{-\frac{\epsilon_2}{2}\Delta N}\right)^2 - \frac{\epsilon_2}{2}\Delta N\right]$$

$\Delta N = \mathcal{R}$



[Tomberg, 2304.10903]



[Tomberg, 2304.10903]

Unreliable tail

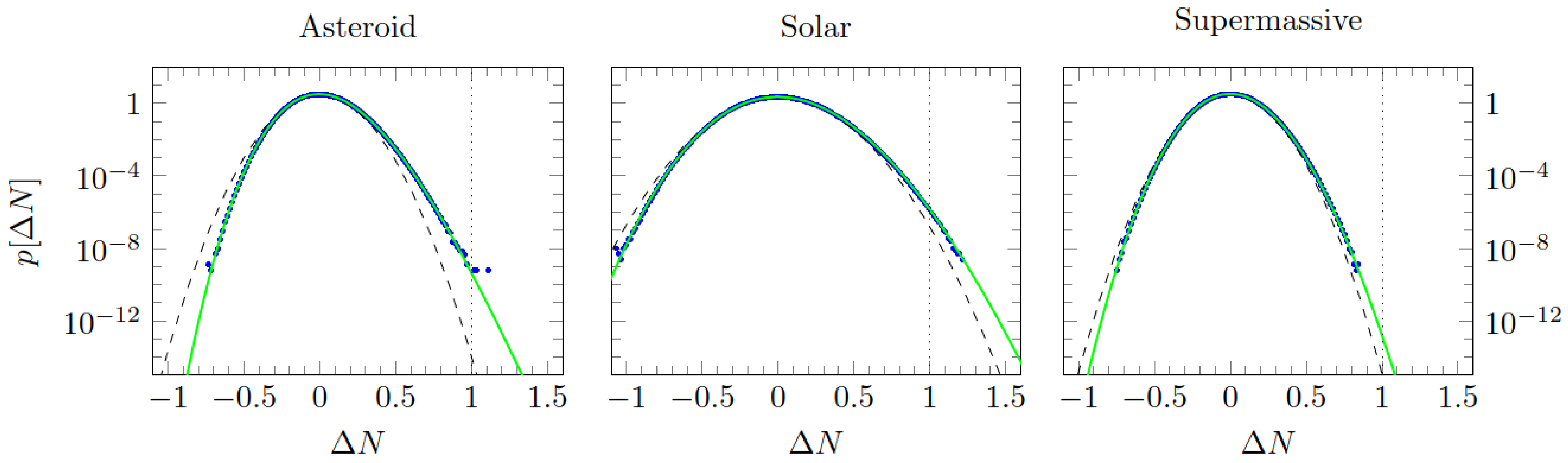
In the tail, the field approaches end of classical trajectory

$$\phi = \frac{2}{\epsilon_2}\pi + \phi_0 = (1 - e^{\frac{\epsilon_2}{2}N})\phi_0$$

Analysis breaks down (field out of CR), when

$$\epsilon_2 \Delta N \gtrsim 2 \ln \frac{2}{\sigma \epsilon_2}$$

[Tomberg, 2304.10903]



Black hole statistics

Beyond collapse threshold in $\mathcal{R}$: compaction function

Stochastic trajectories give detailed knowledge of perturbation profile

Correlations between different scales? Clustering?

Comparison to non-stochastic ΔN

[Cai et al, 1712.09998]
[Biagetti et al, 2105.07810]
[Pi et al, 2211.13932]
…

Same result without stochasticity:

- Compute "total field perturbation" $\Delta\phi$
- Convert to ΔN using classical background eom

"One initial kick"

Works in constant-roll due to linearity of background eom

Conclusions

Stochastic inflation introduces non-Gaussian corrections to PBH statistics

Constant-roll inflation is at the heart of PBH inflation models

$$p(\mathcal{R}) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{2}{\sigma^2\epsilon_2^2}\left(1 - e^{-\frac{\epsilon_2}{2}\mathcal{R}}\right)^2 - \frac{\epsilon_2}{2}\mathcal{R}\right]$$

[Karam et al, 2205.13540]