

Stochastic inflation and primordial black holes: challenges and solutions

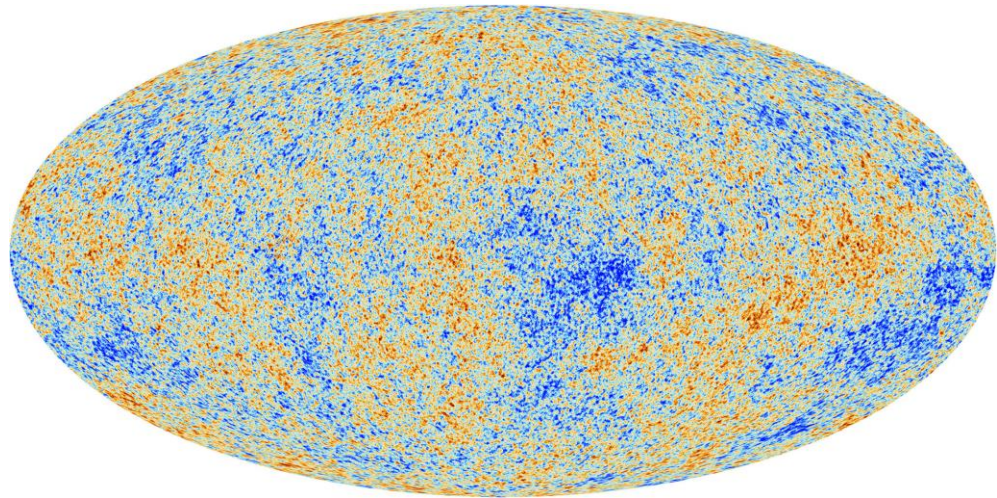
Cosmology from Home 2023
Eemeli Tomberg, NICPB Tallinn

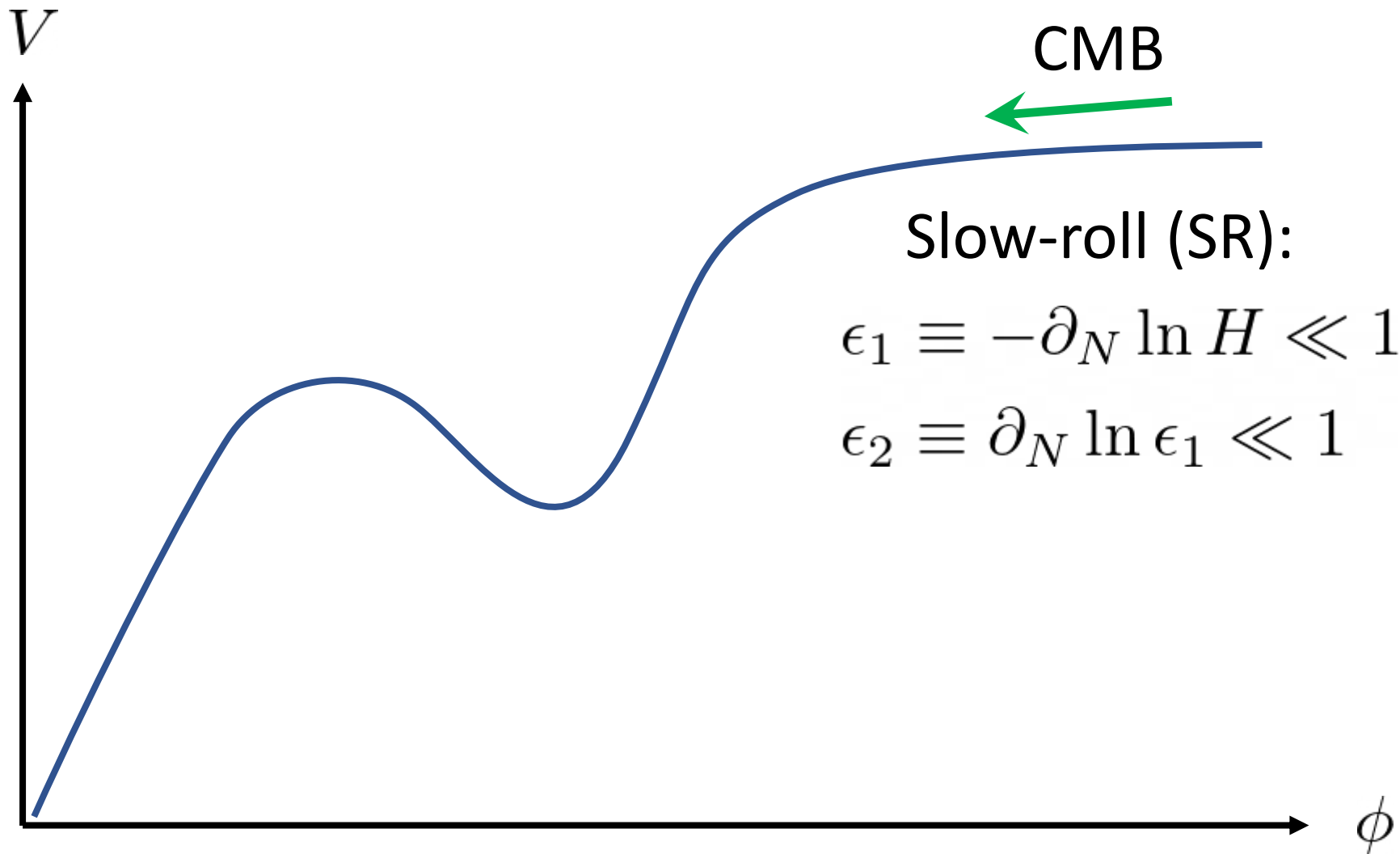
Based on 2012.06551, 2111.07437, 2210.17441, 2304.10903
in collaboration with D. Figueroa, S. Raatikainen, S. Räsänen

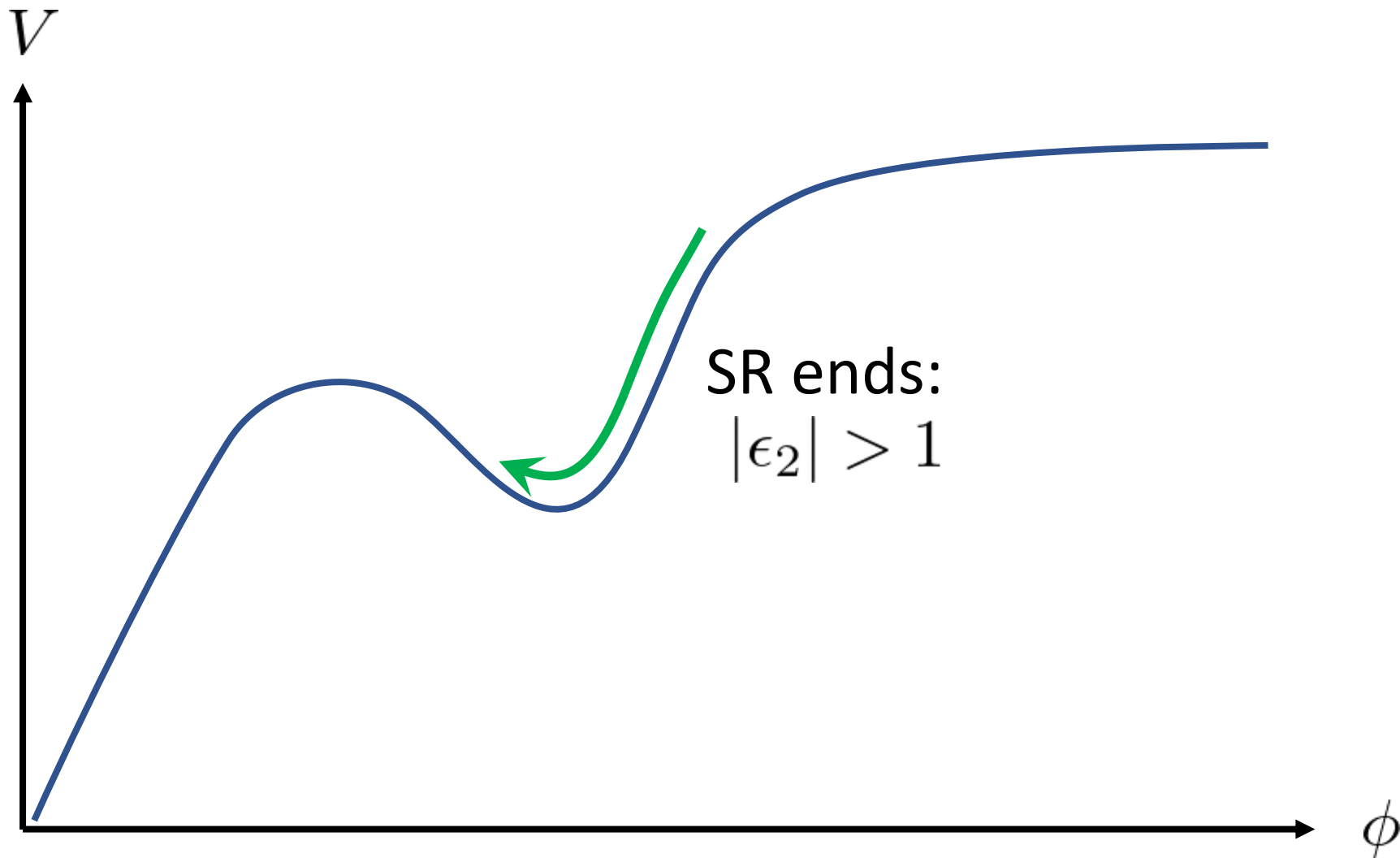
Primordial perturbations

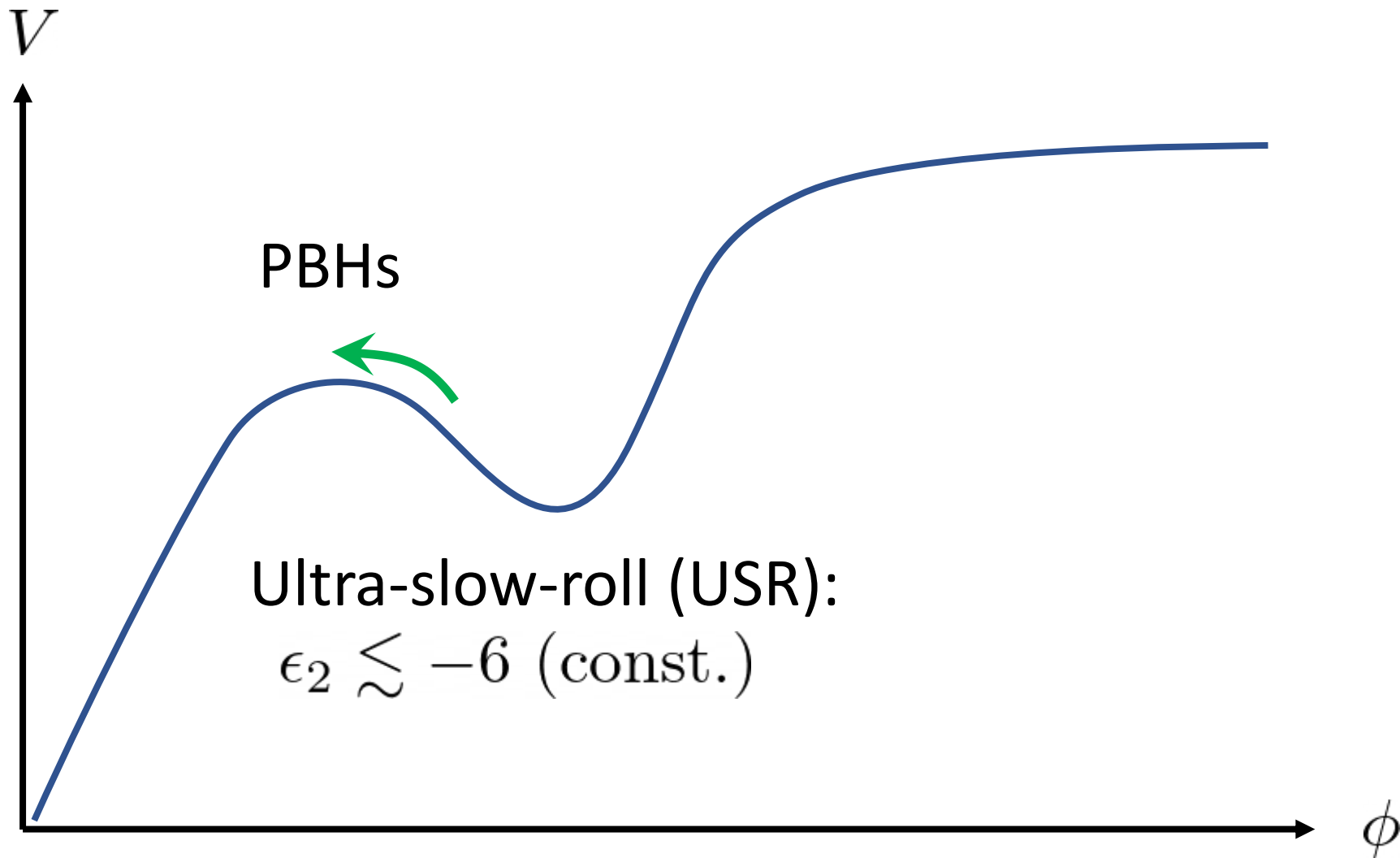
Cosmic inflation: quantum fluctuations

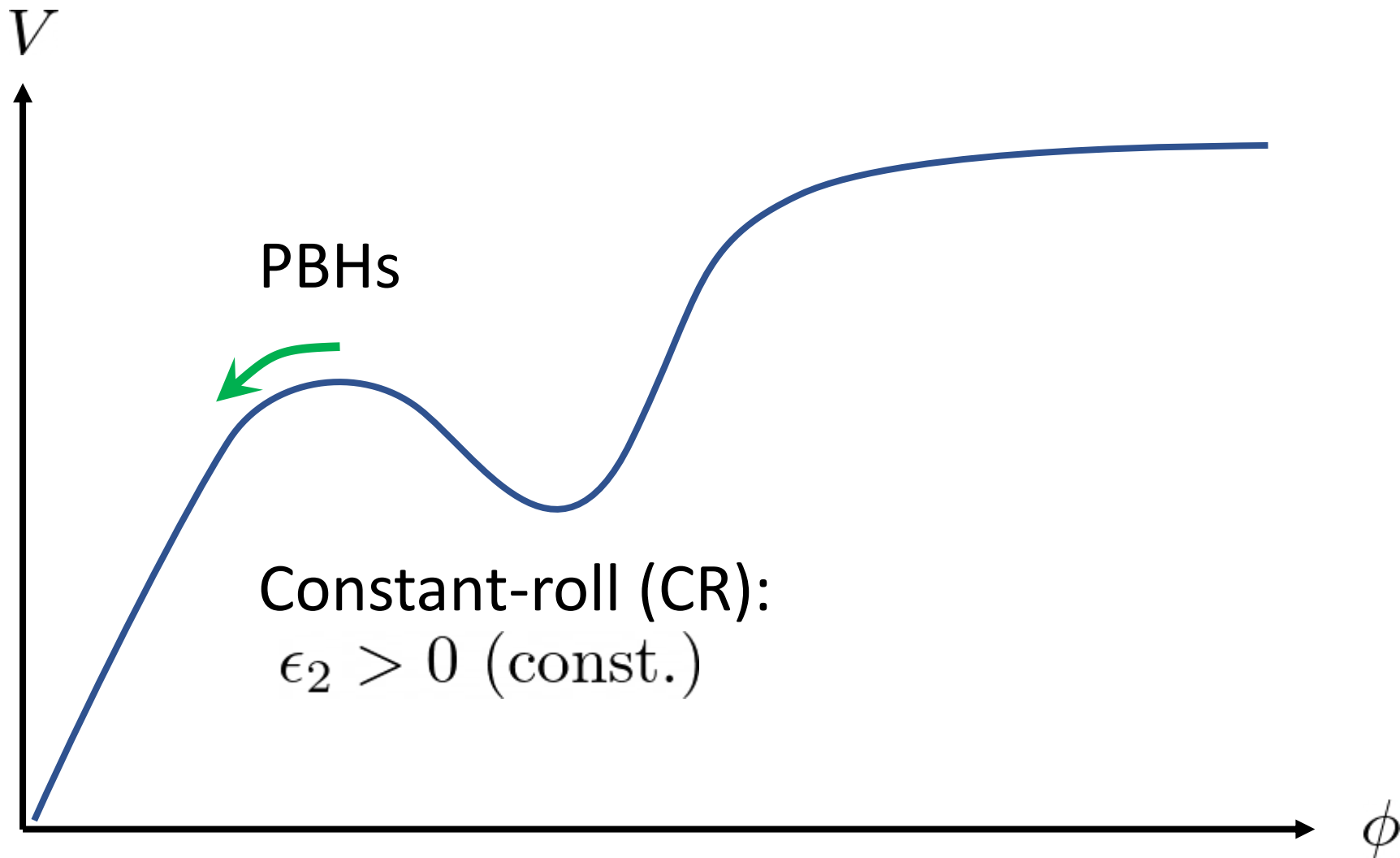
Later: strongest collapse into black holes



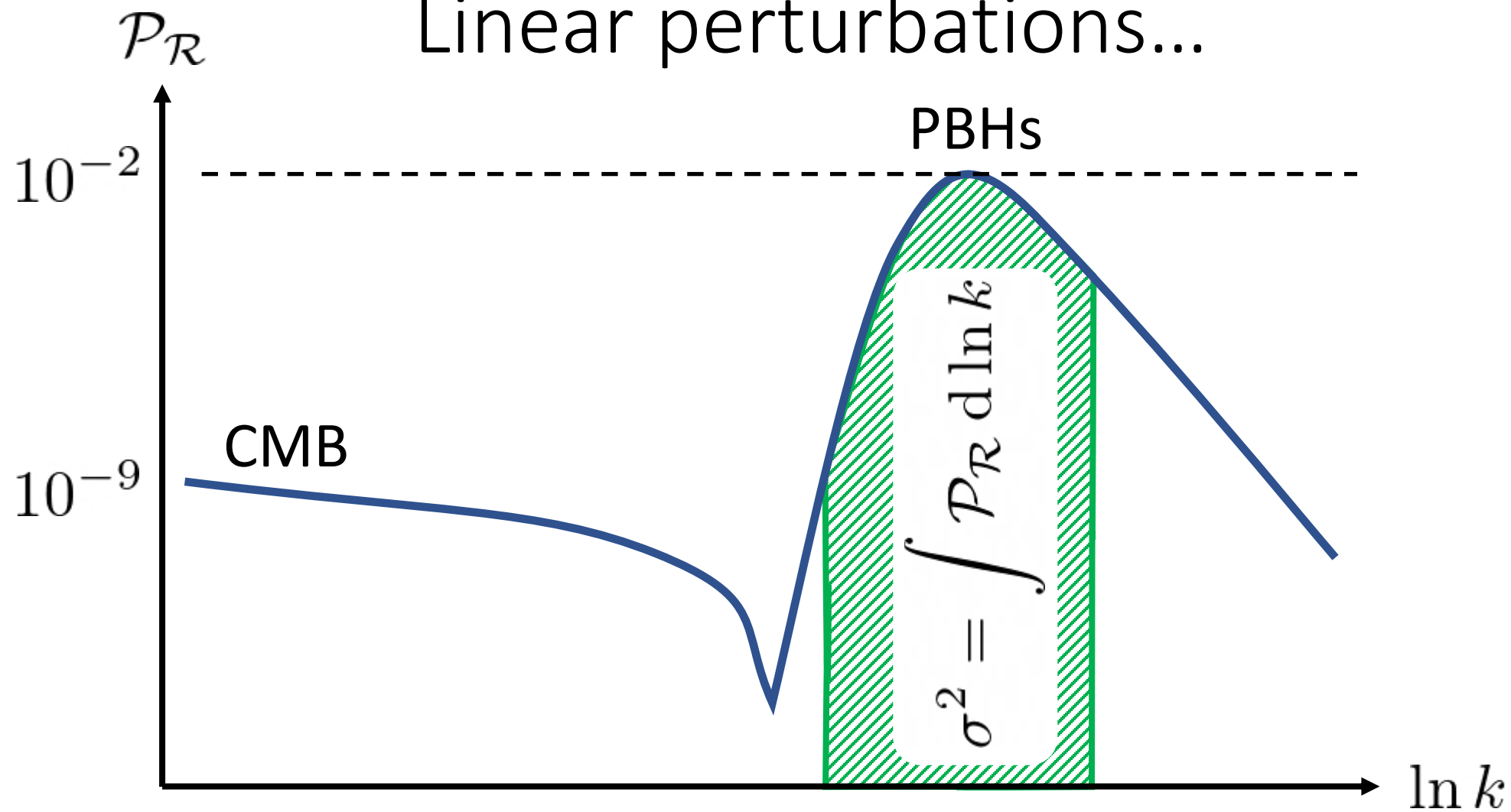




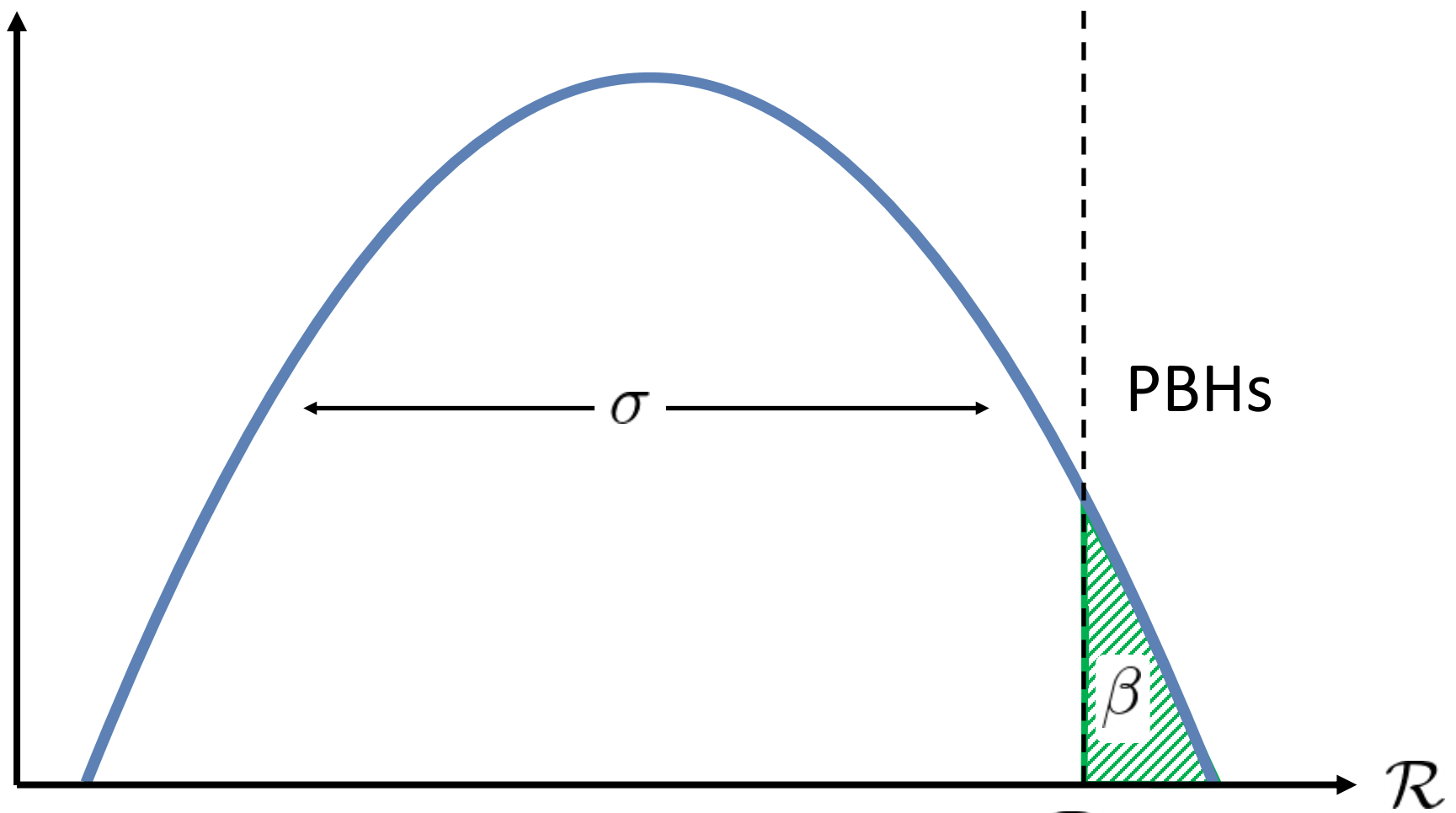




Linear perturbations...



$\log p(\mathcal{R})$...Gaussian distribution



$$p(\mathcal{R}) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\mathcal{R}^2}{2\sigma^2}}$$

Beyond perturbation theory

FLRW evolution is non-linear

Super-Hubble patches \approx local FLRW universes

$$\mathcal{R} = \Delta N$$


Stochastic inflation

Inflaton field: $\phi + \delta\phi$

Coarse-grained:
FLRW



Short-wavelength:
linear perturbation theory



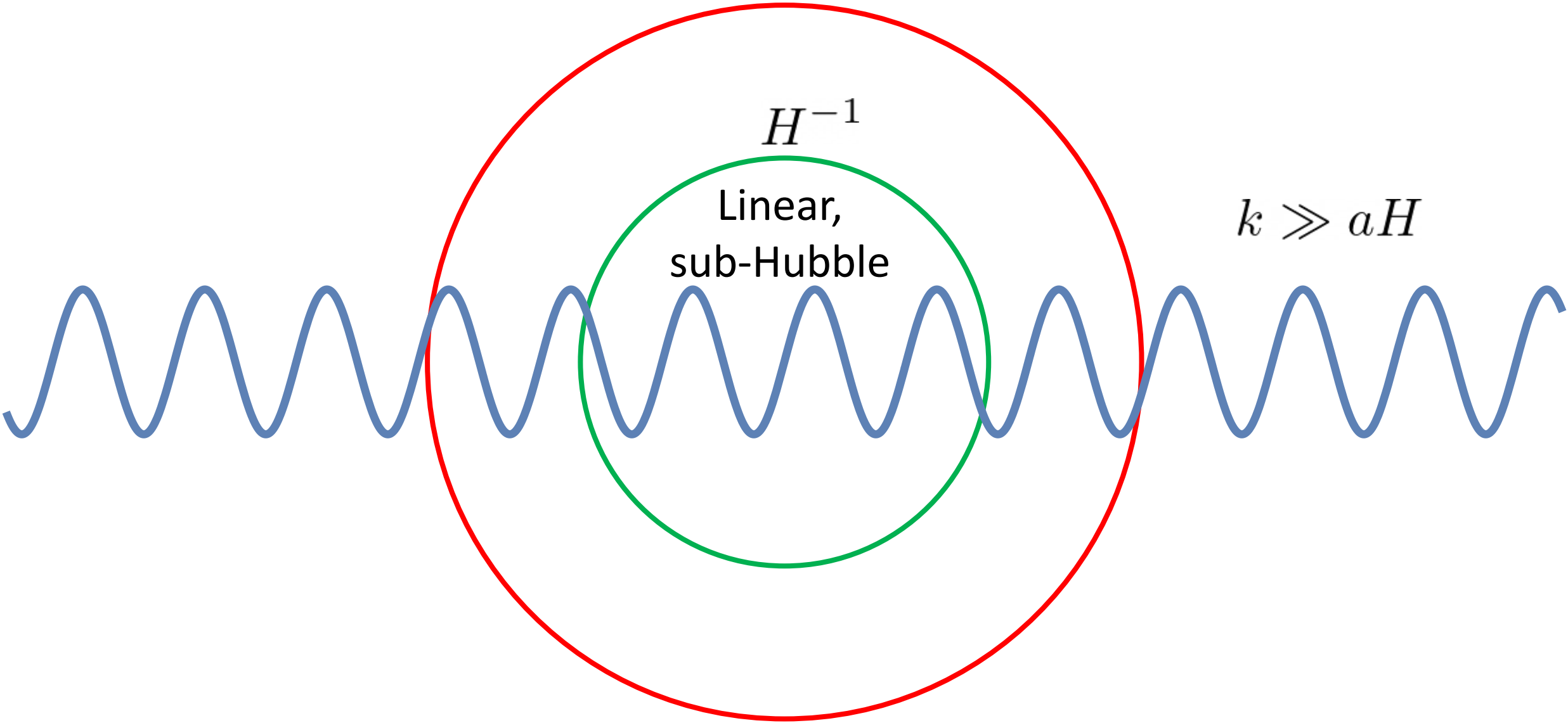
Patched together at the coarse-graining scale $k = k_\sigma \equiv \sigma aH$

$$(\sigma H)^{-1}$$

$$H^{-1}$$

Linear,
sub-Hubble

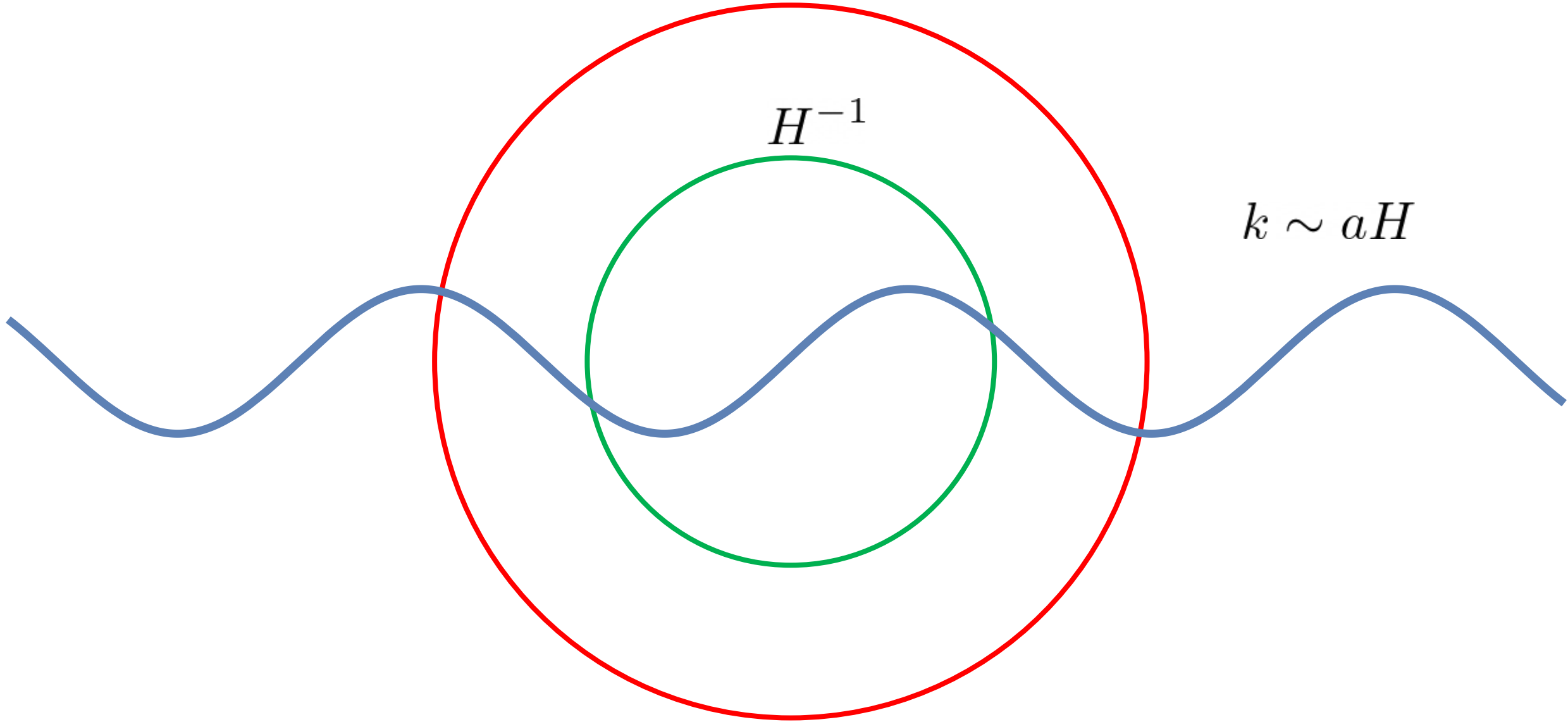
$$k \gg aH$$



$$(\sigma H)^{-1}$$

$$H^{-1}$$

$$k \sim aH$$

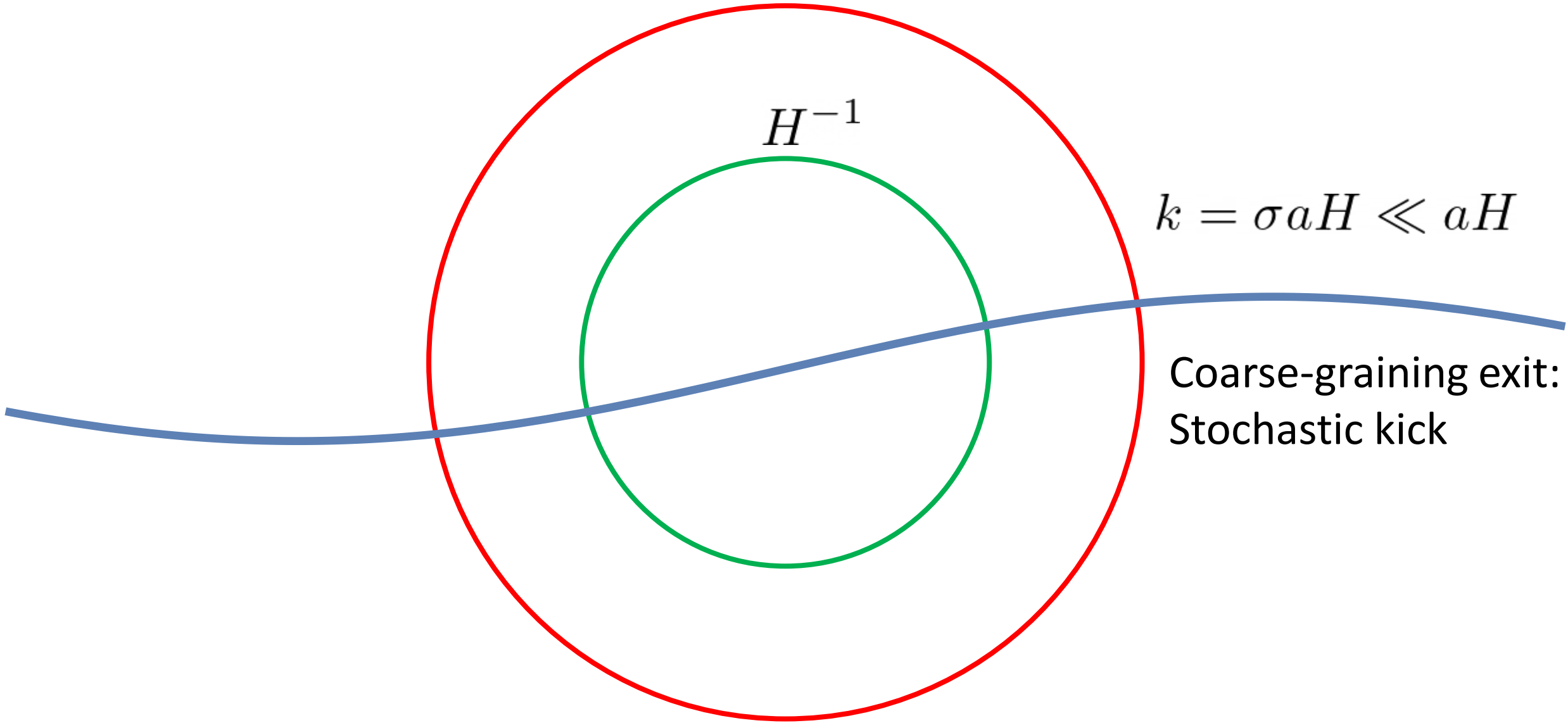


$$(\sigma H)^{-1}$$

$$H^{-1}$$

$$k = \sigma a H \ll a H$$

Coarse-graining exit:
Stochastic kick



Stochastic inflation

$$\phi' = \pi + \xi_\phi, \quad \pi' = - \left(3 - \frac{1}{2}\pi^2 \right) \pi - \frac{V'(\phi)}{H^2} + \xi_\pi, \quad H^2 = \frac{V(\phi)}{3 - \frac{1}{2}\pi^2}$$
$$\delta\phi_k'' = - \left(3 - \frac{1}{2}\pi^2 \right) \delta\phi_k' - \left[\frac{k^2}{a^2 H^2} + \pi^2 \left(3 - \frac{1}{2}\pi^2 \right) + 2\pi \frac{V'(\phi)}{H^2} + \frac{V''(\phi)}{H^2} \right] \delta\phi_k$$

$$\langle \xi_\phi(N) \xi_\phi(N') \rangle = \frac{1}{6\pi^2} \frac{dk_\sigma^3}{dN} |\delta\phi_{k_\sigma}(N)|^2 \delta(N - N')$$

$$\langle \xi_\pi(N) \xi_\pi(N') \rangle = \frac{1}{6\pi^2} \frac{dk_\sigma^3}{dN} |\delta\phi'_{k_\sigma}(N)|^2 \delta(N - N')$$

$$\langle \xi_\phi(N) \xi_\pi(N') \rangle = \frac{1}{6\pi^2} \frac{dk_\sigma^3}{dN} \delta\phi_{k_\sigma}(N) \delta\phi'_{k_\sigma}^*(N) \delta(N - N')$$

$$\mathcal{R} = \Delta N \equiv N - \bar{N}$$

Challenges

Need to solve a coupled set of complicated equations

Need statistics: PBHs from rare realizations

Need accuracy: perturbations sensitive to small changes

Further approximations?

[Pattison et al, 1707.00537]

[Ezquiaga et al, 1805.06731]

[Biagetti et al, 1804.07124]

...

Assume de Sitter behavior for $\delta\phi$: $|\delta\phi_{k_\sigma}|^2 \approx \frac{H^2}{2k_\sigma^3}$

Assume a simplified potential (flat sections, steps, ...)

Get analytical solutions. Accuracy?

Numerics?

[Figueroa et al, 2012.06551]

[Figueroa et al, 2111.07437]

Solve with supercomputers (millions of CPU hours)

Only case studies: tuning parameters too costly

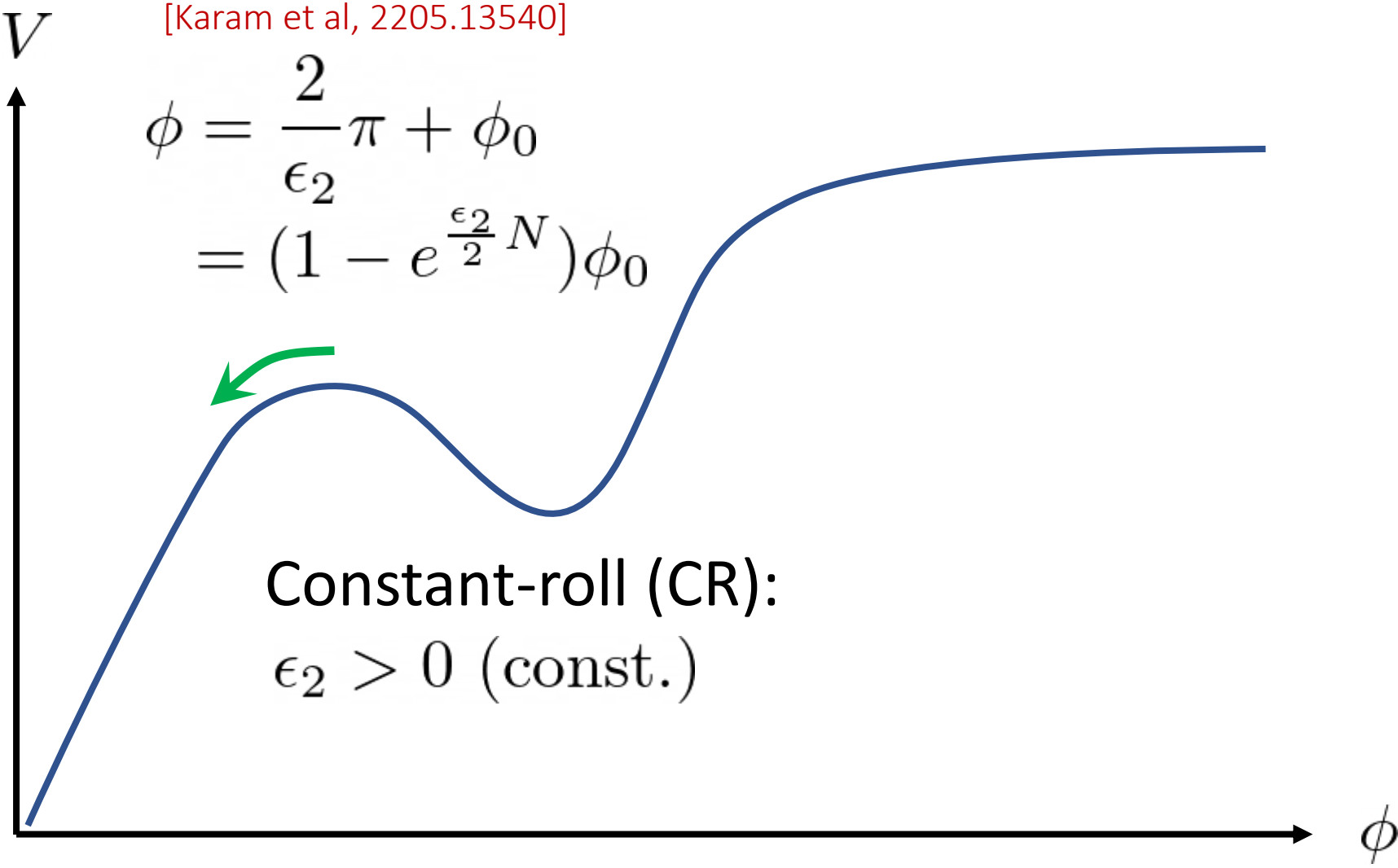
Solution: constant-roll [Tomberg, 2304.10903]

- CR: coarse-graining of strongest modes
- Power spectrum frozen: motion confined to classical trajectory
- Classical trajectory simple

Can solve the system analytically,
for any linear power spectrum

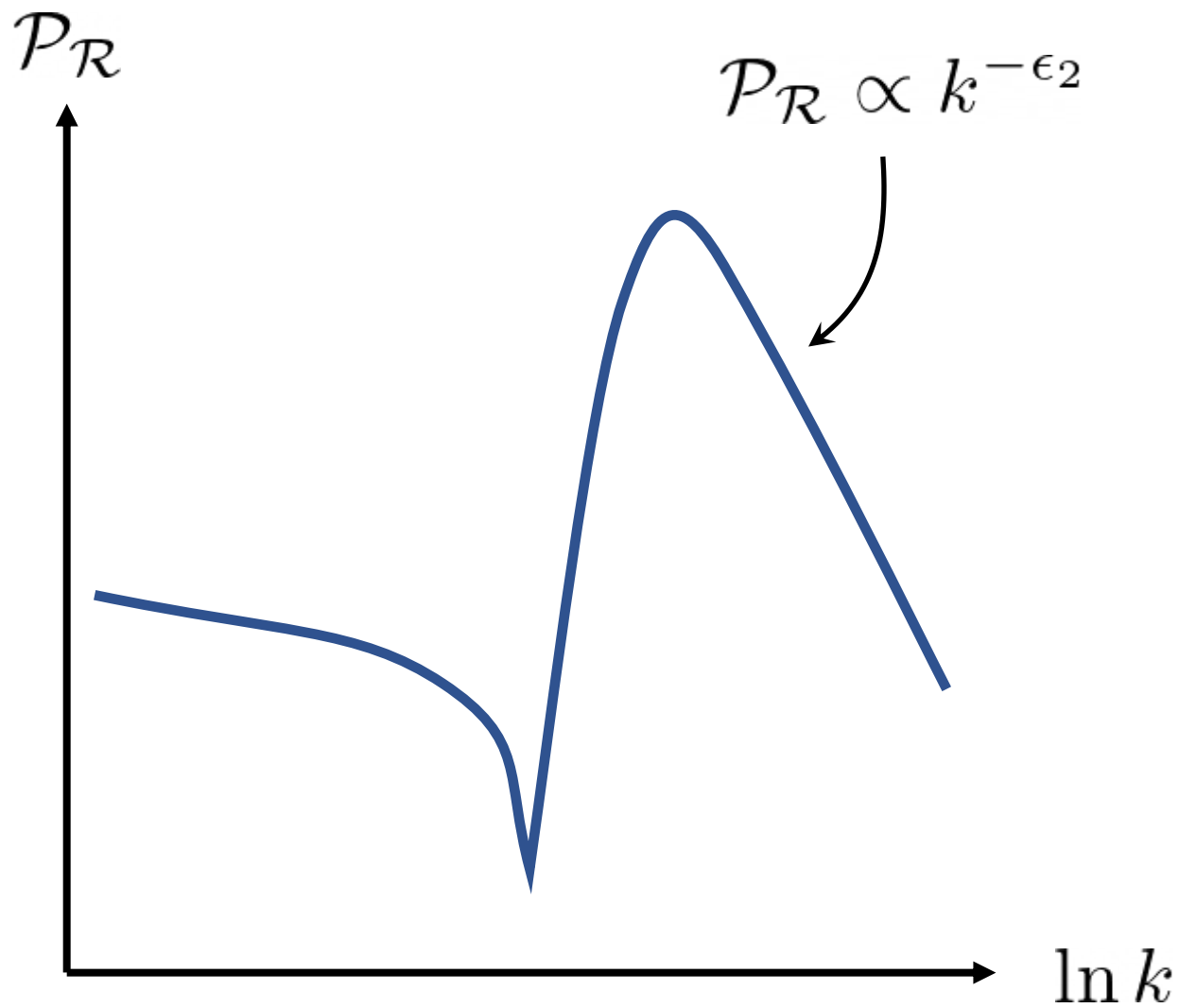
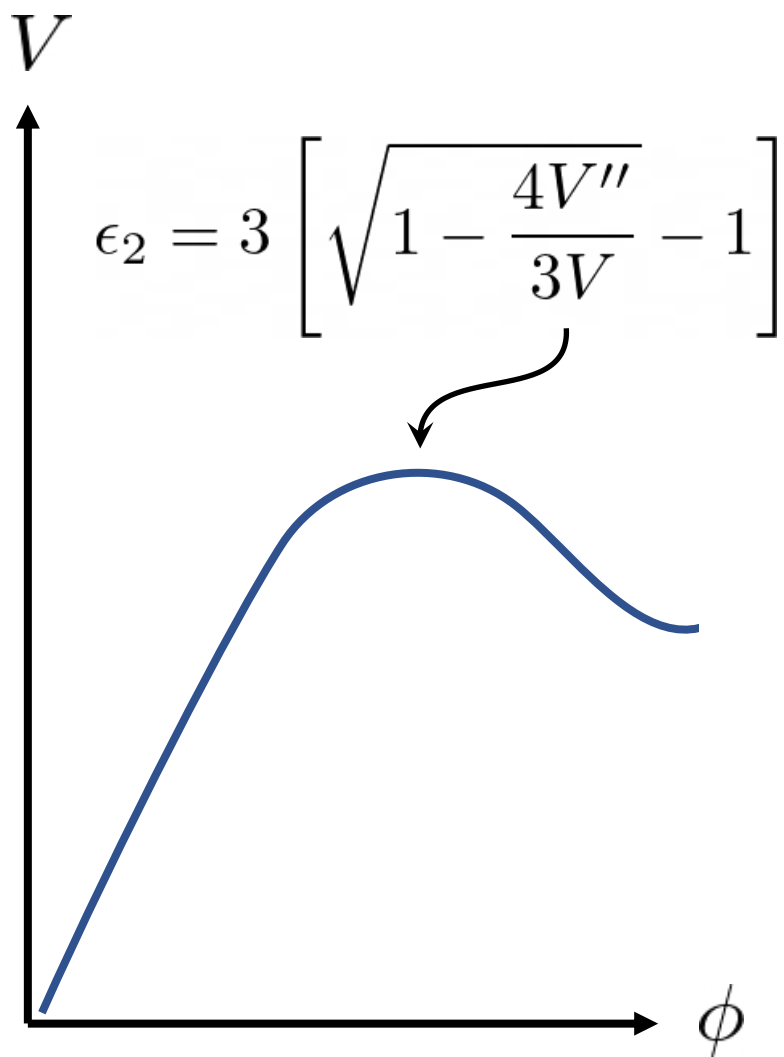
[Karam et al, 2205.13540]

$$\begin{aligned}\phi &= \frac{2}{\epsilon_2} \pi + \phi_0 \\ &= (1 - e^{\frac{\epsilon_2}{2} N}) \phi_0\end{aligned}$$



Constant-roll (CR):
 $\epsilon_2 > 0$ (const.)


[Karam et al, 2205.13540]



Simplified stochastic equation:

$$d\phi = \frac{\epsilon_2}{2}(\phi - \phi_0)dN + \frac{\epsilon_2}{2}\phi_0 e^{\frac{\epsilon_2}{2}N} \sqrt{\mathcal{P}_{\mathcal{R}}(k_\sigma)}dN \hat{\xi}_N$$

$$\phi(N) = \phi_0 \left(1 - e^{\frac{\epsilon_2}{2}N}\right) + \frac{\epsilon_2}{2}\phi_0 e^{\frac{\epsilon_2}{2}N} X(N)$$


$$\langle \hat{\xi}_N \hat{\xi}_{N'} \rangle = \delta_{NN'}$$

$$X(N) \equiv \sum_{k=k_{\text{ini}}}^{k=k_\sigma(N)} \sqrt{\mathcal{P}_{\mathcal{R}}(k)} d \ln k \hat{\xi}_k$$

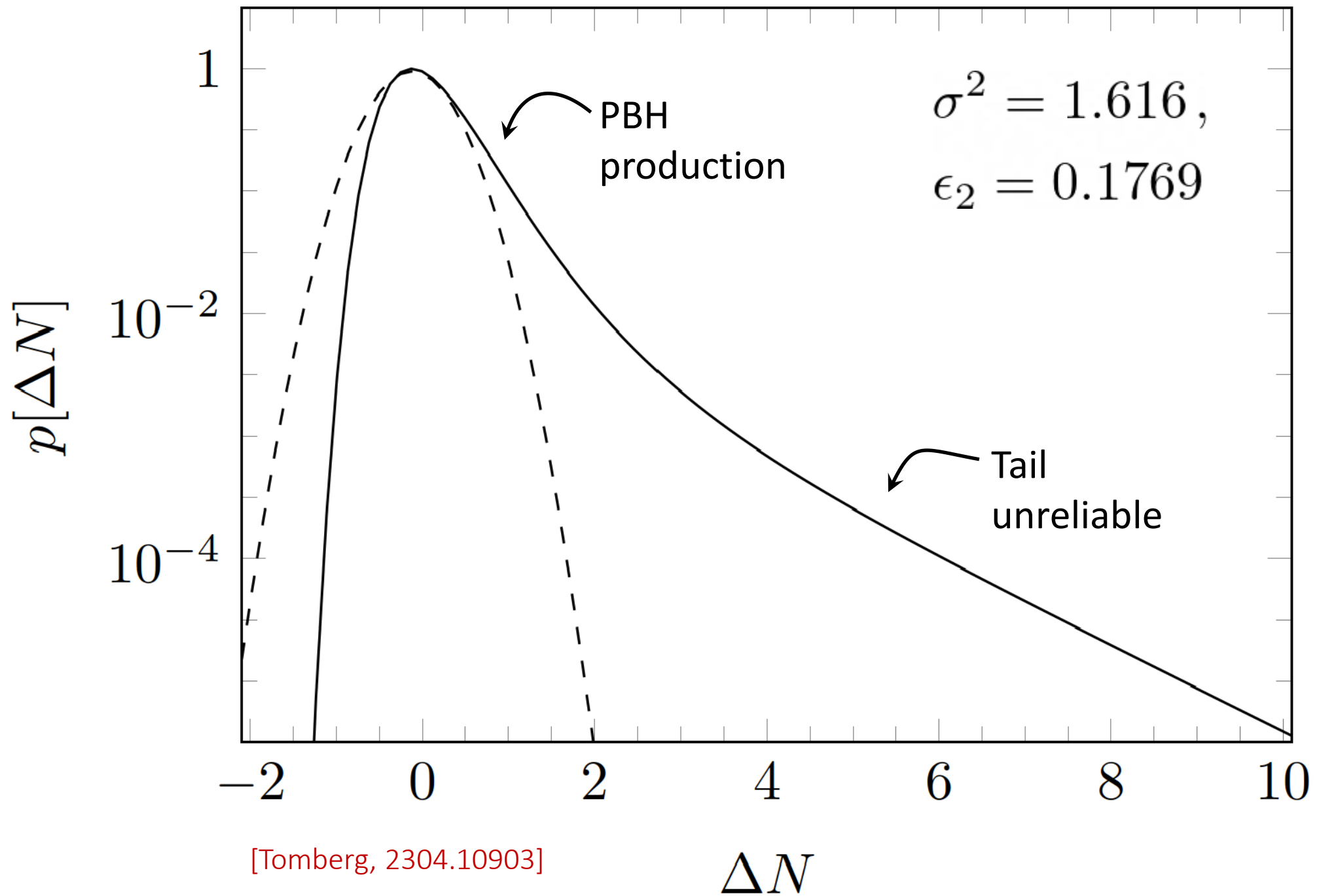
ΔN distribution

$$p(X) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{X^2}{2\sigma^2}}, \quad \sigma^2 \equiv \int_{k_{\text{ini}}}^{k_{\text{end}}} \mathcal{P}_{\mathcal{R}}(k) d \ln k$$

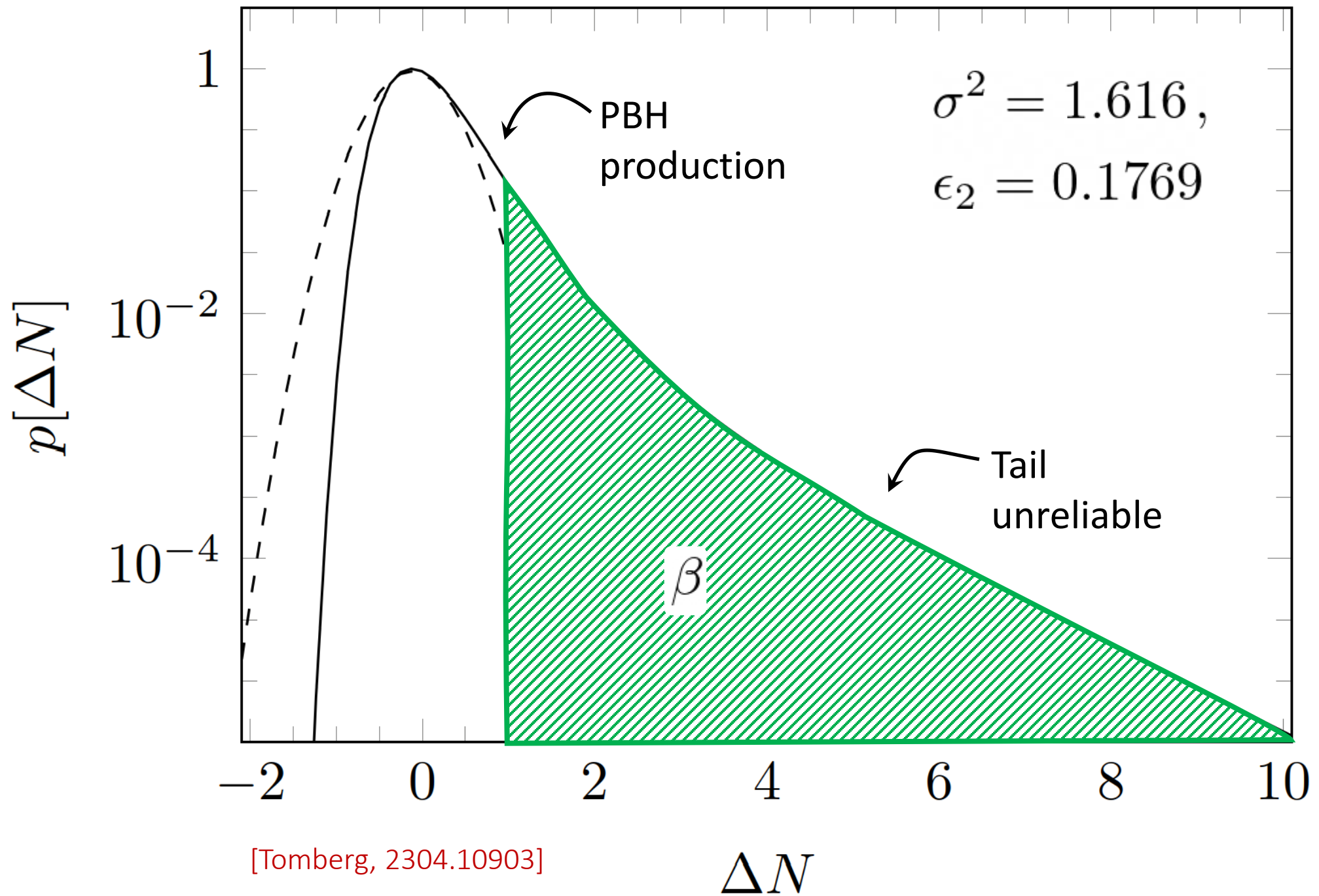
$$X = \frac{2}{\epsilon_2} \left(1 - e^{-\frac{\epsilon_2}{2} \Delta N} \right)$$

$$p(\Delta N) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left[-\frac{2}{\sigma^2 \epsilon_2^2} \left(1 - e^{-\frac{\epsilon_2}{2} \Delta N} \right)^2 - \frac{\epsilon_2}{2} \Delta N \right]$$

$\Delta N = \mathcal{R}$



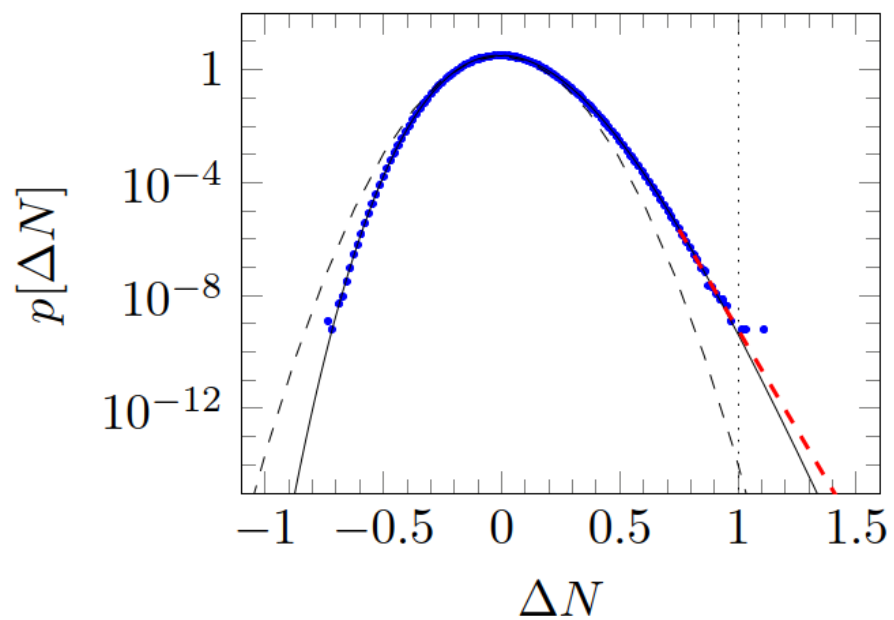
[Tomberg, 2304.10903]



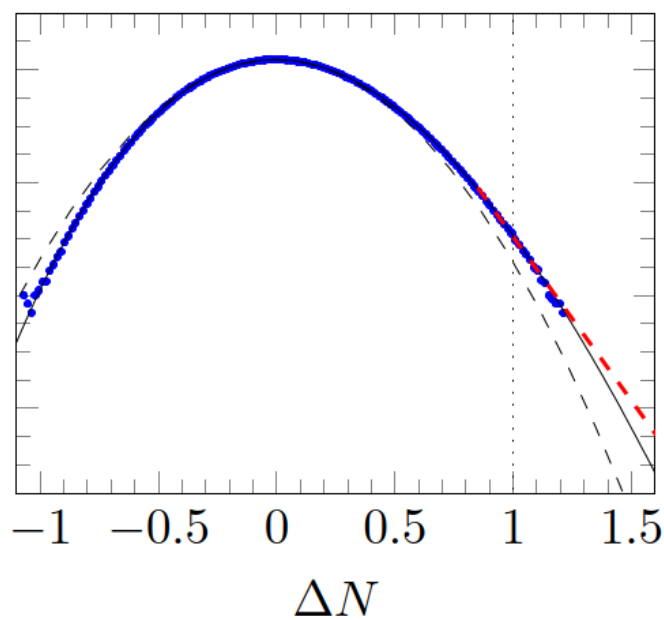
[Tomberg, 2304.10903]

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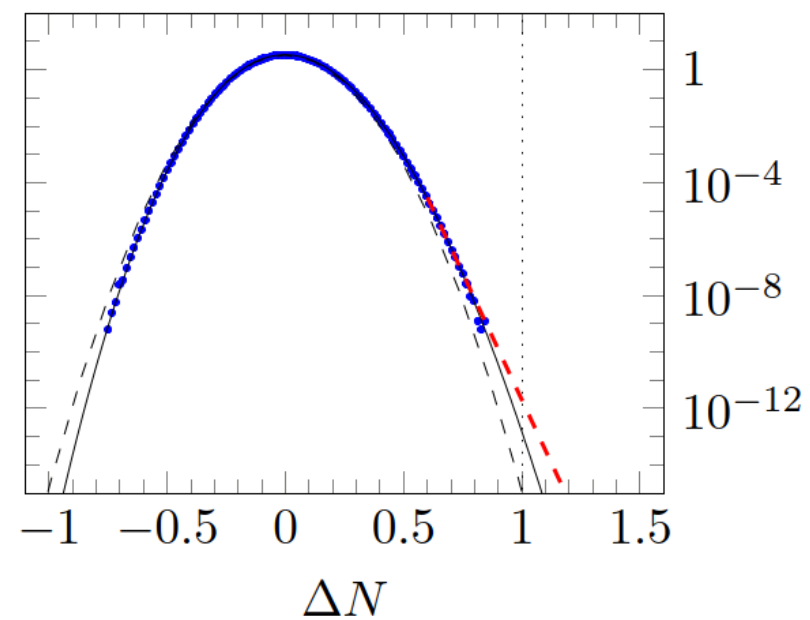
Asteroid



Solar



Supermassive



Black = Constant-roll approximation

Dashed = Gaussian fit

Blue = numerical computation

Red = numerical extrapolation

Comparison to non-stochastic ΔN [Cai et al, 1712.09998]
[Biagetti et al, 2105.07810]
[Pi et al, 2211.13932]
...

Same result without stochasticity:

- Compute “total field perturbation” $\Delta\phi$
- Convert to ΔN using classical background eom

“One initial kick”

Works in constant-roll due to **linearity of background eom**

Conclusions

Stochastic inflation introduces non-Gaussian corrections to PBH statistics

Constant-roll inflation is at the heart of PBH inflation models

$$p(\mathcal{R}) = \frac{1}{\sqrt{2\pi\sigma}} \exp \left[-\frac{2}{\sigma^2 \epsilon_2^2} \left(1 - e^{-\frac{\epsilon_2}{2} \mathcal{R}} \right)^2 - \frac{\epsilon_2}{2} \mathcal{R} \right]$$