Stochastic constant-roll inflation: a tool to compute primordial black hole statistics

COSMO’23, Madrid, Sep 2023

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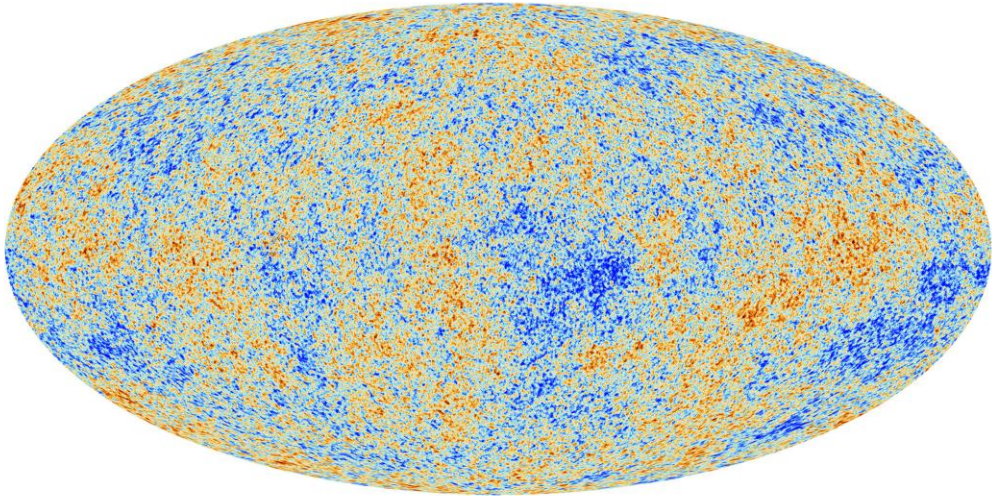
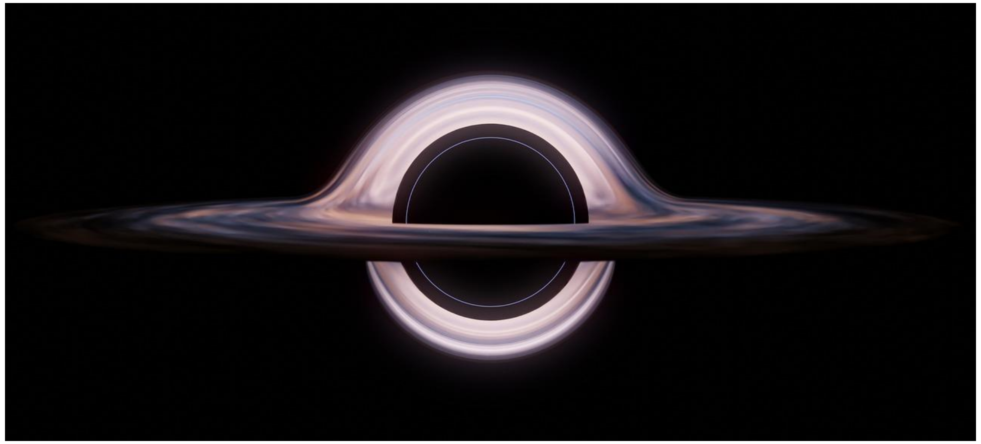
Based on 2012.06551, 2111.07437, 2210.17441, 2304.10903
in collaboration with D. Figueroa, S. Raatikainen, S. Räsänen

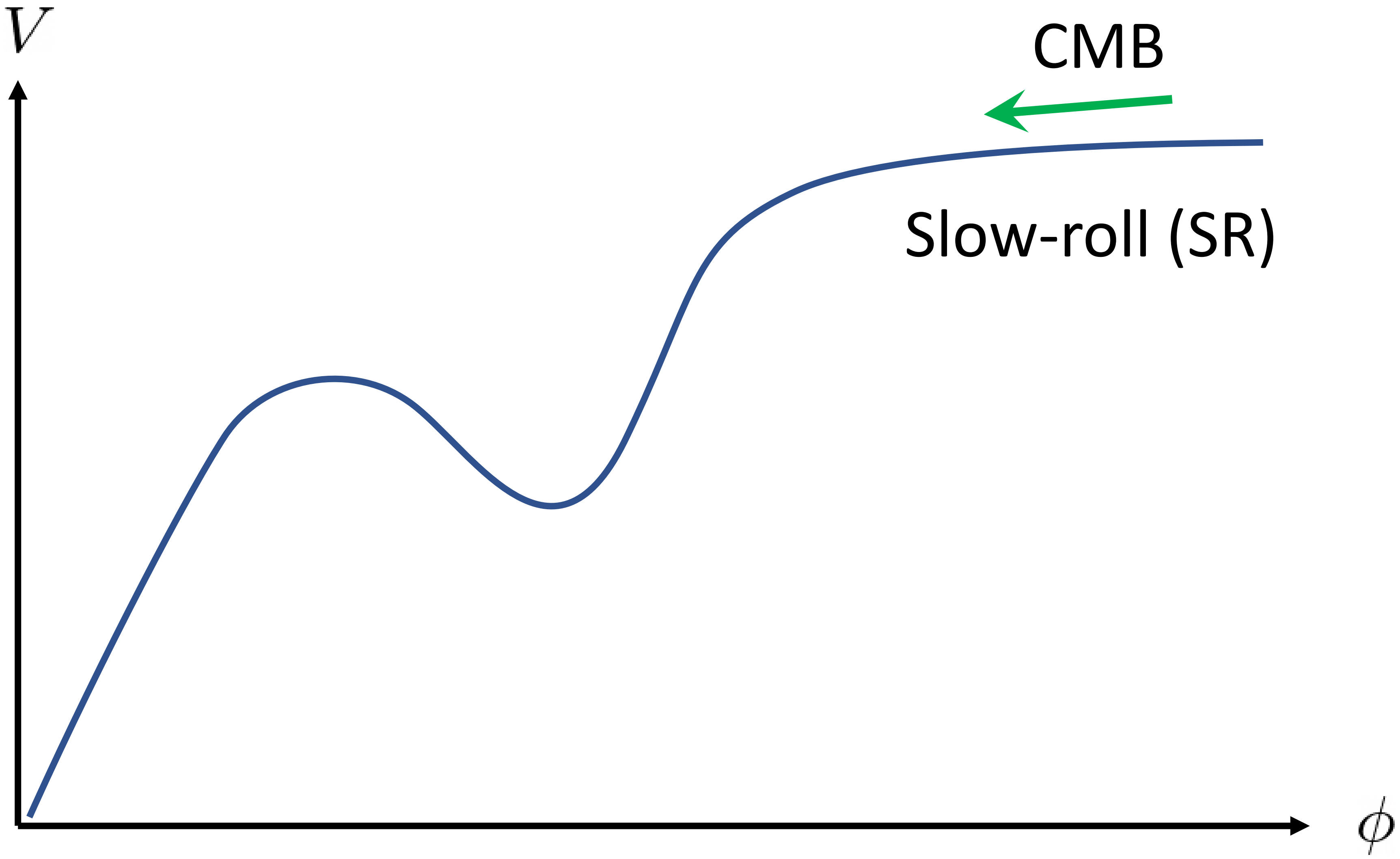
Primordial perturbations

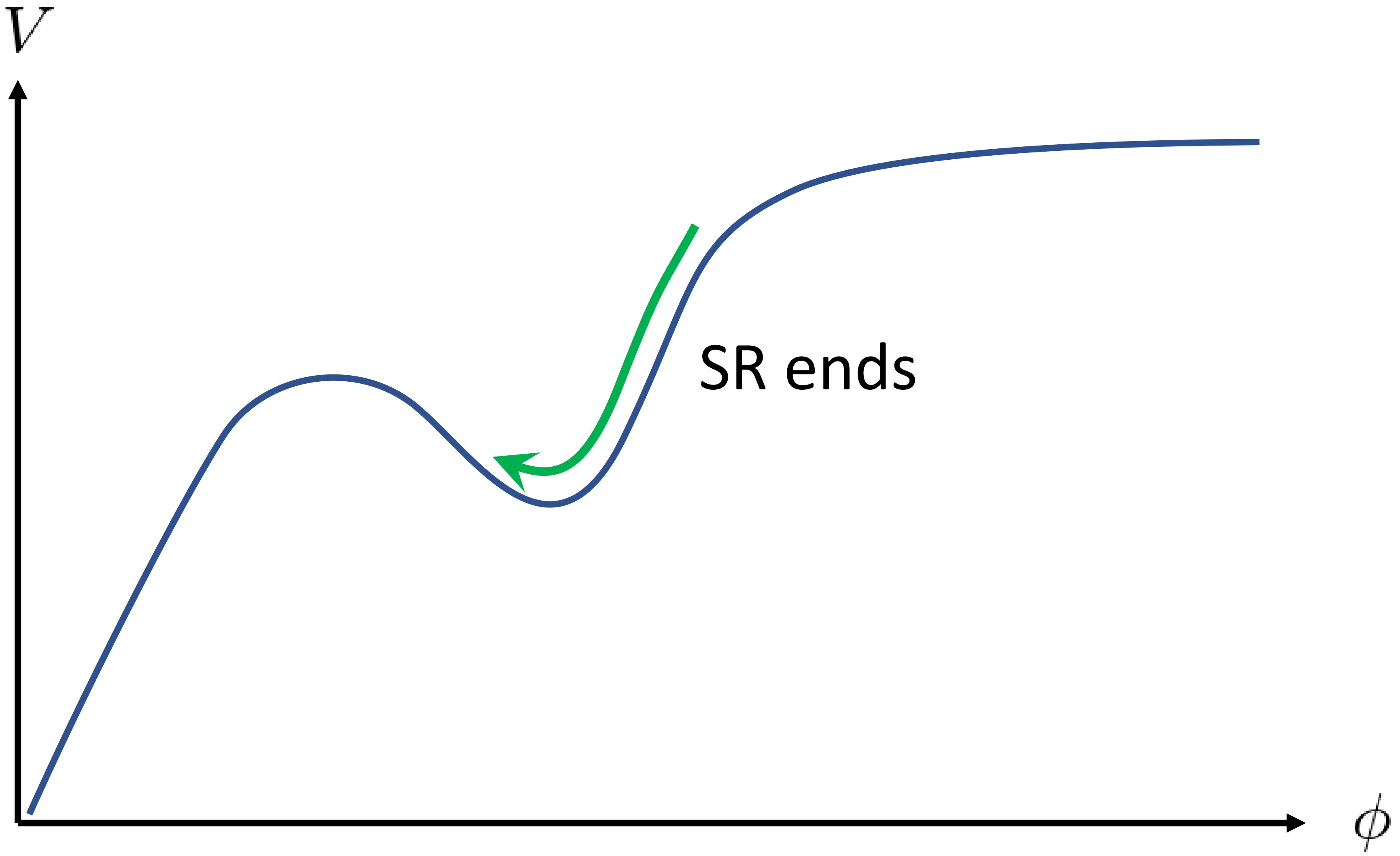
Cosmic inflation: quantum fluctuations

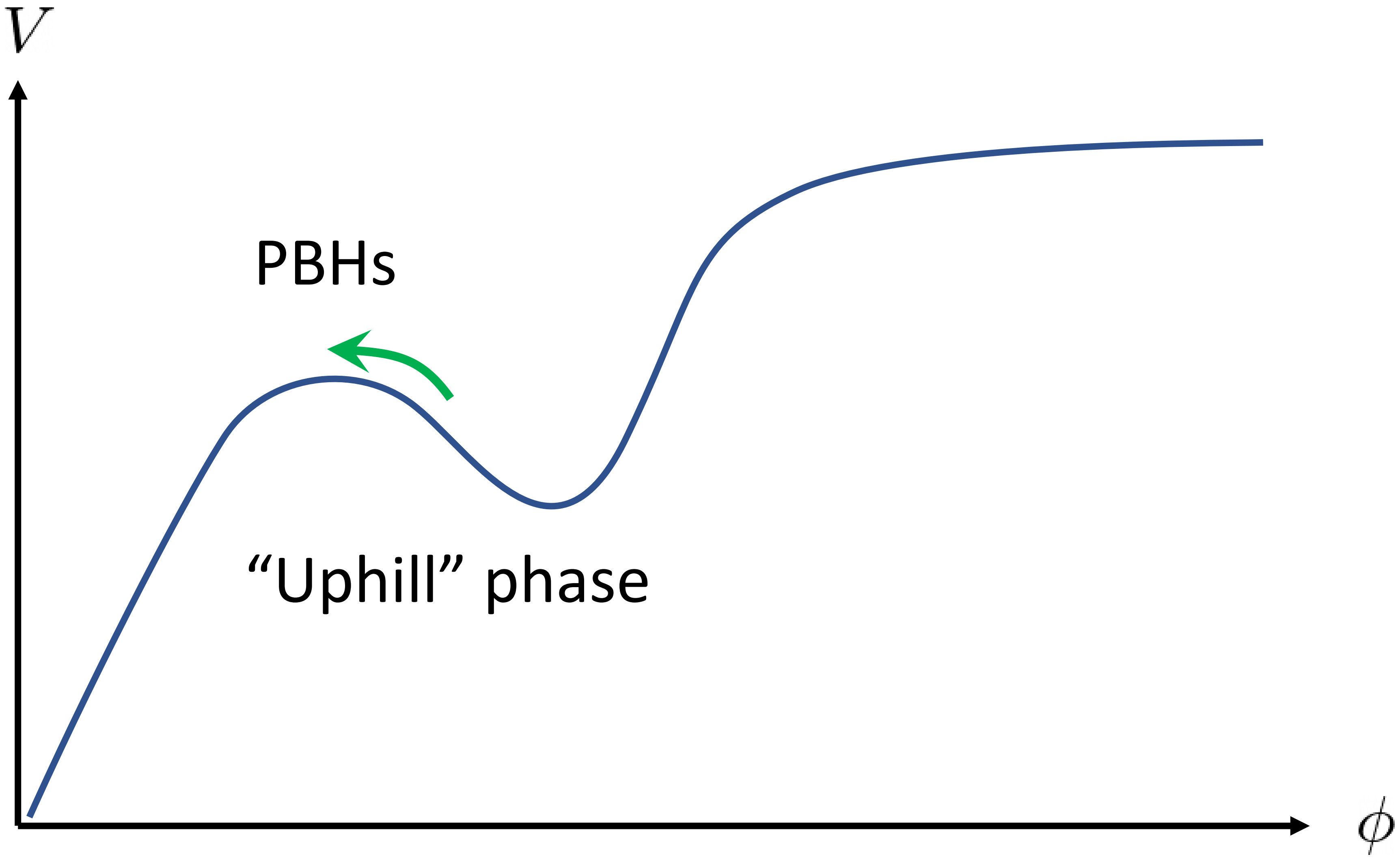
Later: strongest collapse into black holes





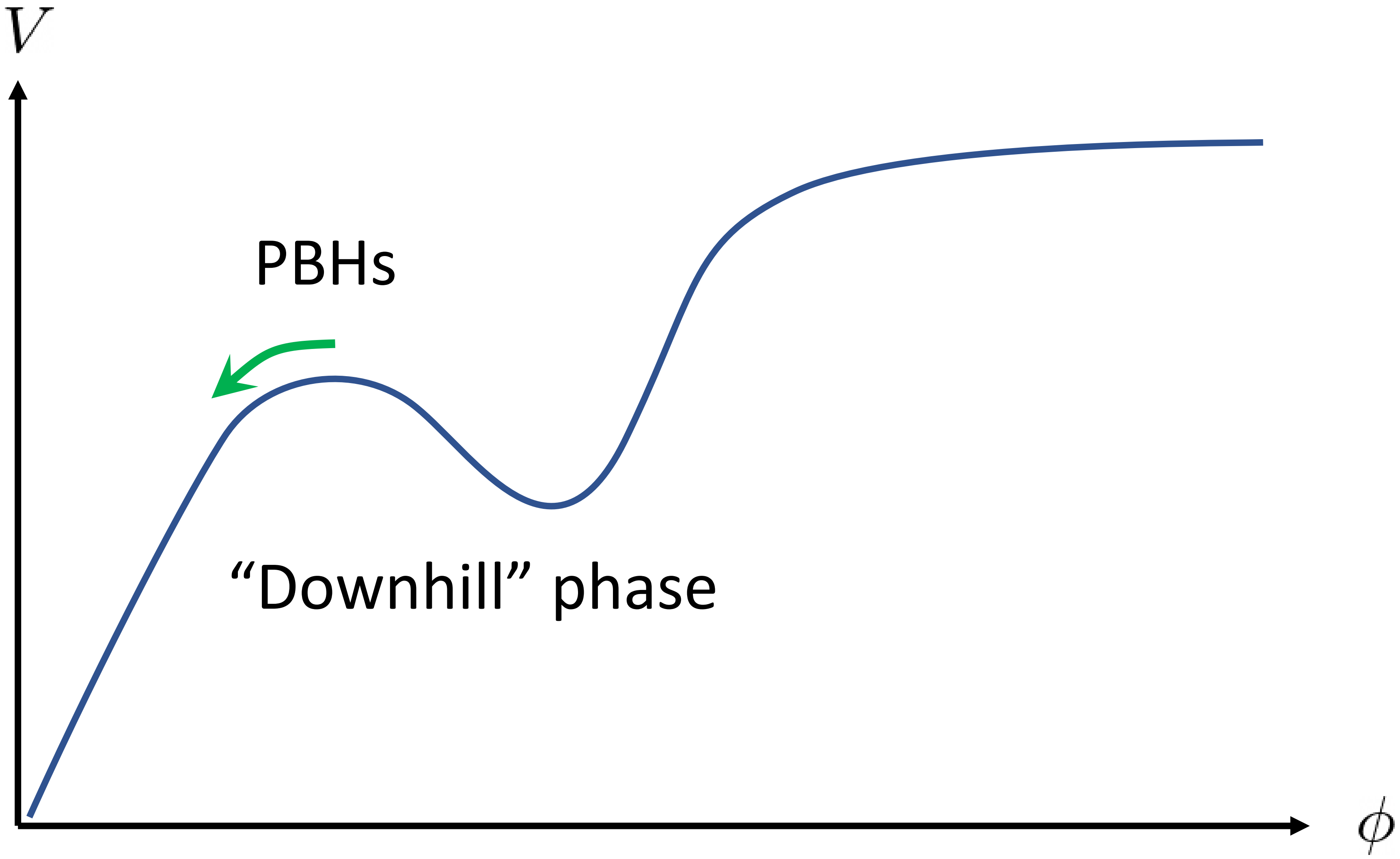

V
CMB
Slow-roll (SR)
ϕ


V
SR ends
ϕ

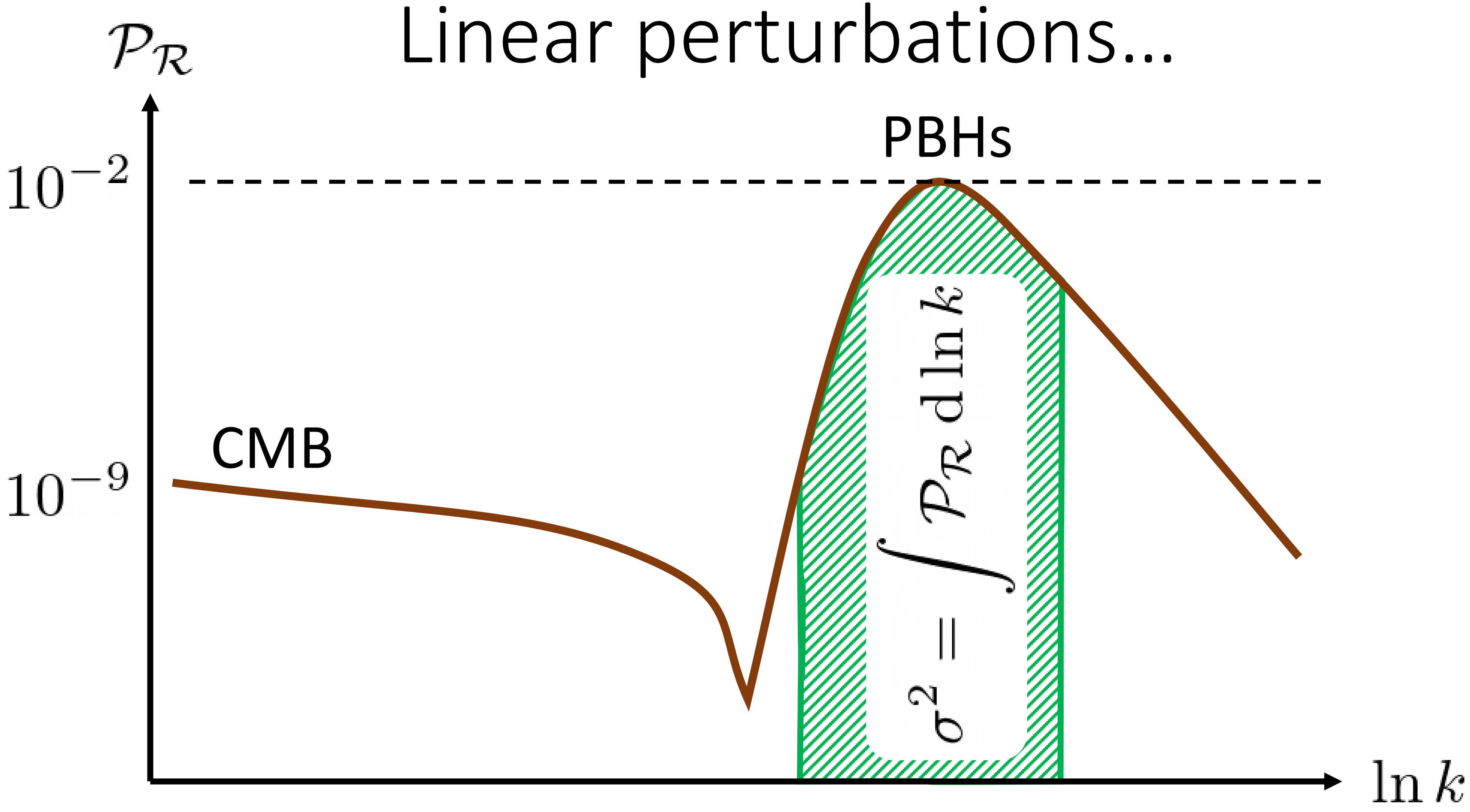


PBHs

“Uphill” phase

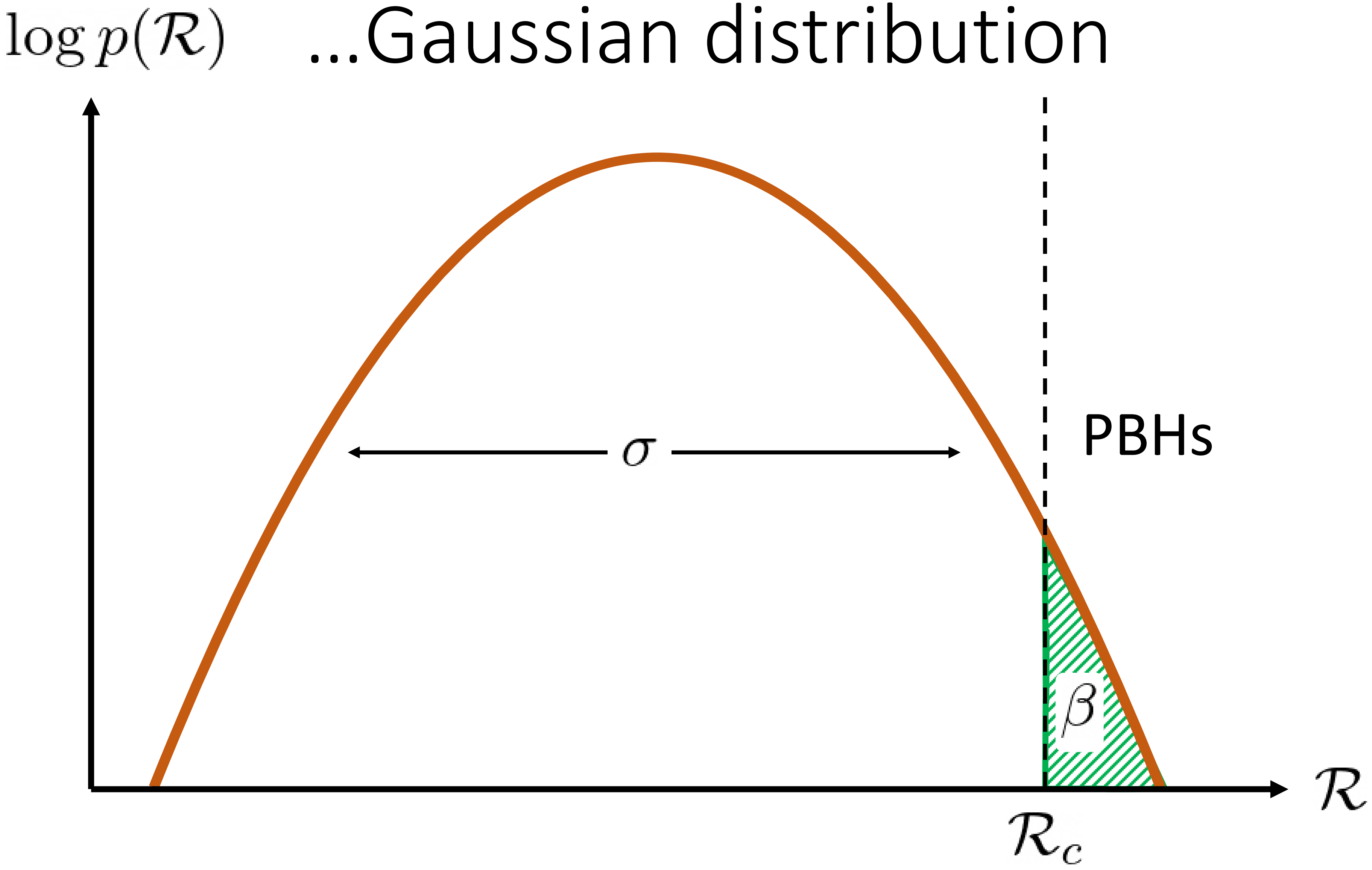

V
PBHs
“Downhill” phase
ϕ

Linear perturbations...



$$\sigma^2 = \int \mathcal{P}_\mathcal{R} \, \mathrm{d} \ln k$$

…Gaussian distribution



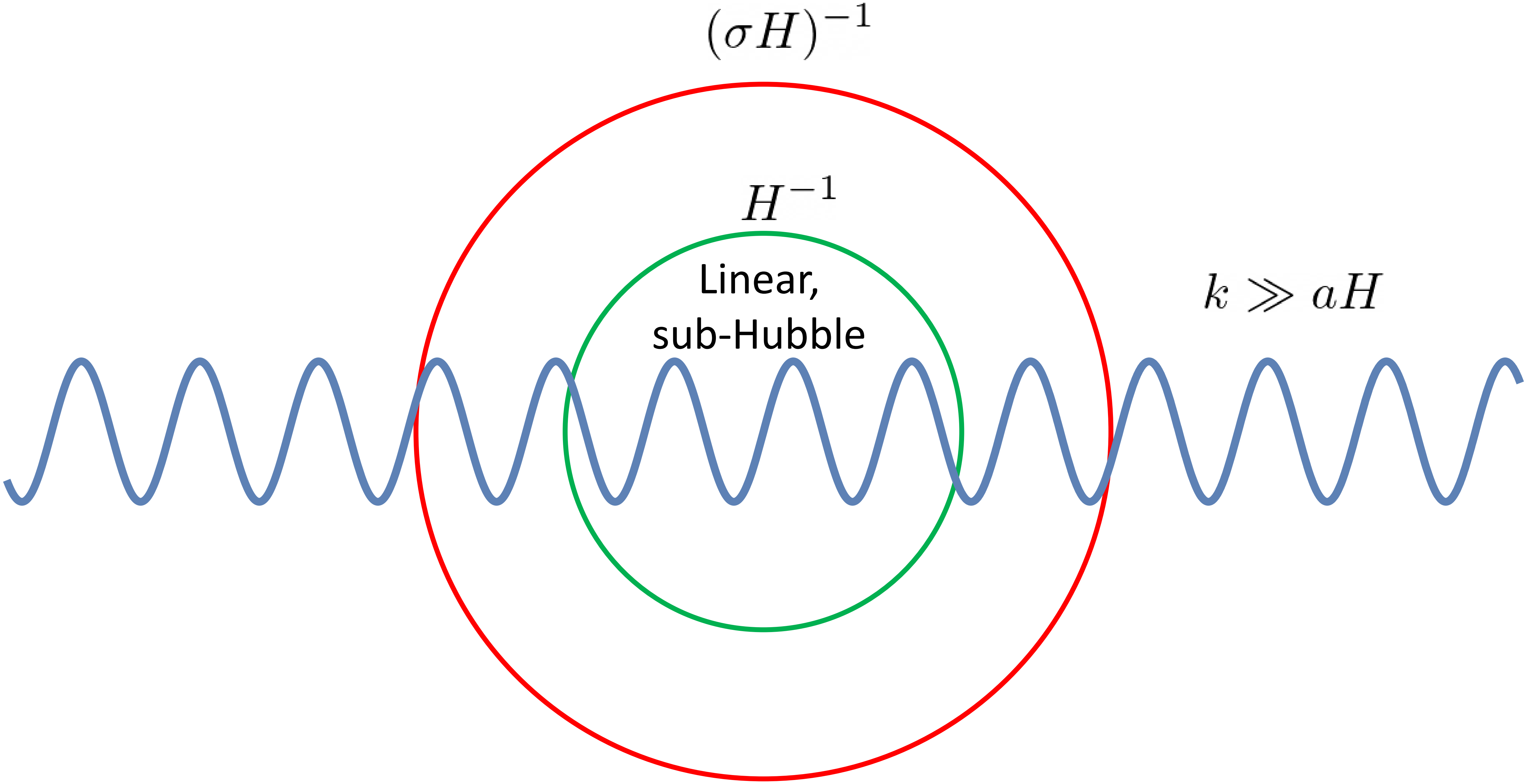
Stochastic inflation

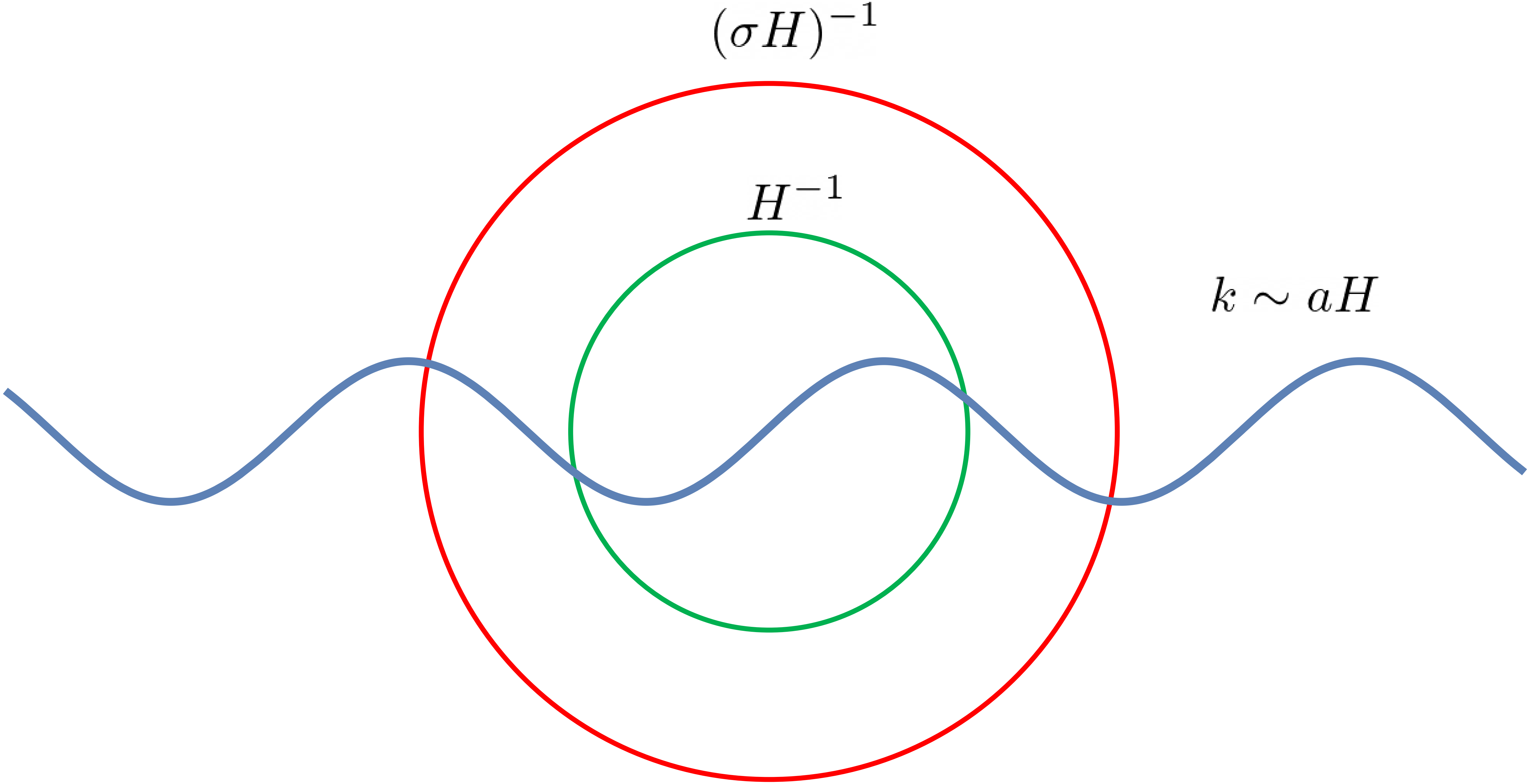
Inflaton field: $\phi + \delta\phi$

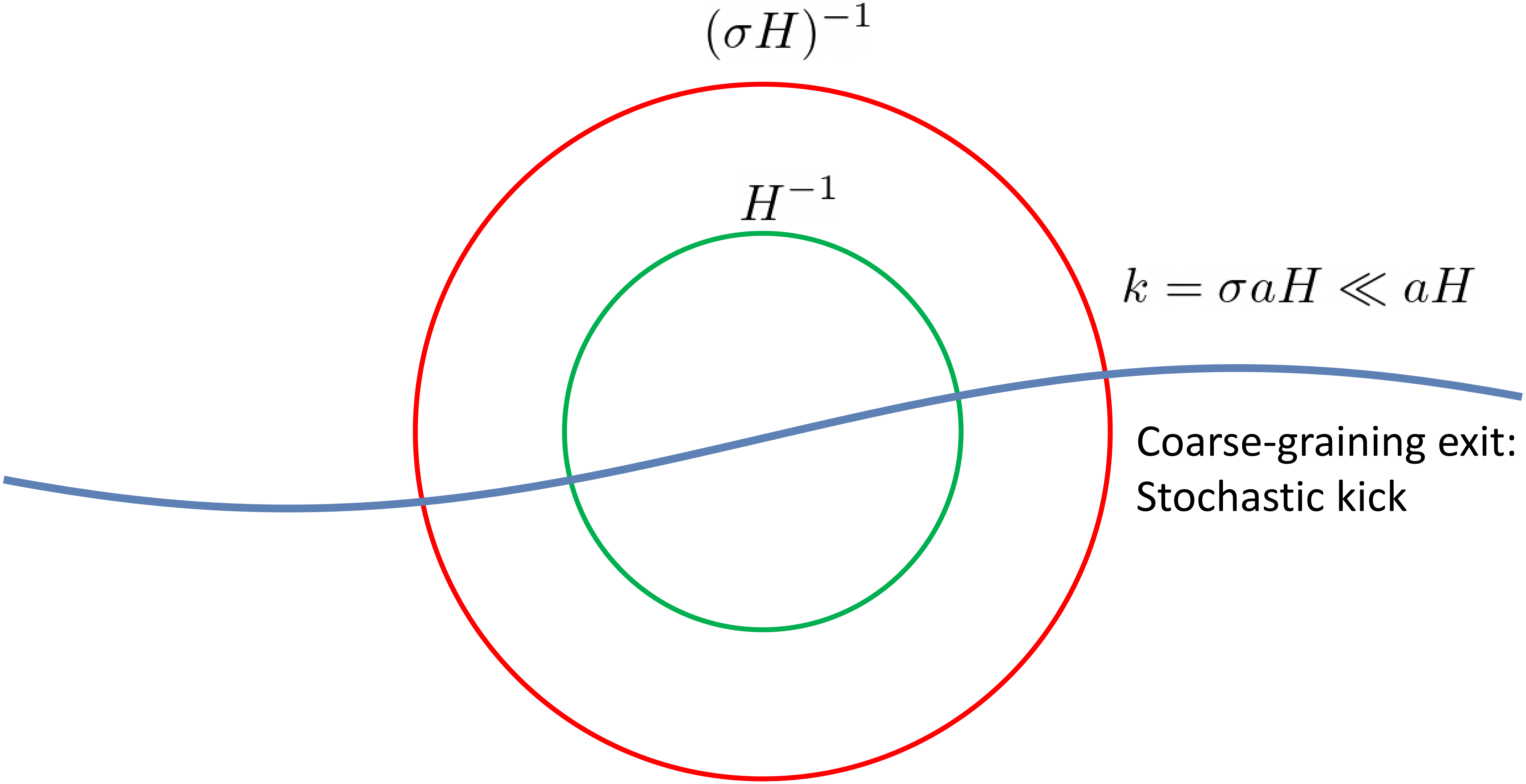
Coarse-grained:
FLRW

Short-wavelength:
linear perturbation theory

Patched together at the coarse-graining scale $k = k_\sigma \equiv \sigma a H$


$(\sigma H)^{-1}$
H^{-1}
Linear,
sub-Hubble
$k \gg aH$


$(\sigma H)^{-1}$
H^{-1}
$k \sim aH$



Coarse-graining exit:
Stochastic kick

Stochastic inflation

$$\phi' = \pi + \xi_\phi\,, \quad \pi' = -\left(3 - \frac{1}{2}\pi^2\right)\pi - \frac{V'(\phi)}{H^2} + \xi_\pi\,, \quad H^2 = \frac{V(\phi)}{3 - \frac{1}{2}\pi^2}$$

$$\delta\phi_k'' = -(3 - \frac{1}{2}\pi^2)\delta\phi_k' - \left[\frac{k^2}{a^2H^2} + \pi^2(3 - \frac{1}{2}\pi^2) + 2\pi\frac{V'(\phi)}{H^2} + \frac{V''(\phi)}{H^2}\right]\delta\phi_k$$

$$\langle\xi_\phi(N)\xi_\phi(N')\rangle = \frac{1}{6\pi^2}\frac{\mathrm{d}k_\sigma^3}{\mathrm{d}N}|\delta\phi_{k_\sigma}(N)|^2\delta(N - N')$$

$$\langle\xi_\pi(N)\xi_\pi(N')\rangle = \frac{1}{6\pi^2}\frac{\mathrm{d}k_\sigma^3}{\mathrm{d}N}|\delta\phi'_{k_\sigma}(N)|^2\delta(N - N')$$

$$\langle\xi_\phi(N)\xi_\pi(N')\rangle = \frac{1}{6\pi^2}\frac{\mathrm{d}k_\sigma^3}{\mathrm{d}N}\delta\phi_{k_\sigma}(N)\delta\phi'^*_{k_\sigma}(N)\delta(N - N')$$

$$\mathcal{R} = \Delta N \equiv N - \bar{N}$$

Stochastic inflation

$\phi' = \pi + \xi_\phi\,, \quad \pi' = -\left(3 - \frac{1}{2}\pi^2\right)\pi - \frac{V'}{}$ [illegible] $H^2 = \frac{V(\phi)}{3 - \frac{1}{2}\pi^2}$

$\delta\phi_k'' = -(3 - \frac{1}{2}\pi^2)\delta\phi_k' - \Big[\frac{k^2}{}$ [illegible] $) + 2\pi\frac{V'(\phi)}{H^2} + \frac{V''(\phi)}{H^2}\Big]\delta\phi_k$

$\langle\xi_\phi(N)\xi_\phi(N')\rangle = \frac{1}{}$ [illegible] $\delta(N - N')$

$\langle\xi_\pi(N)\xi_\pi(N'$ [illegible] $_{k_\sigma}(N)|^2\delta(N - N')$

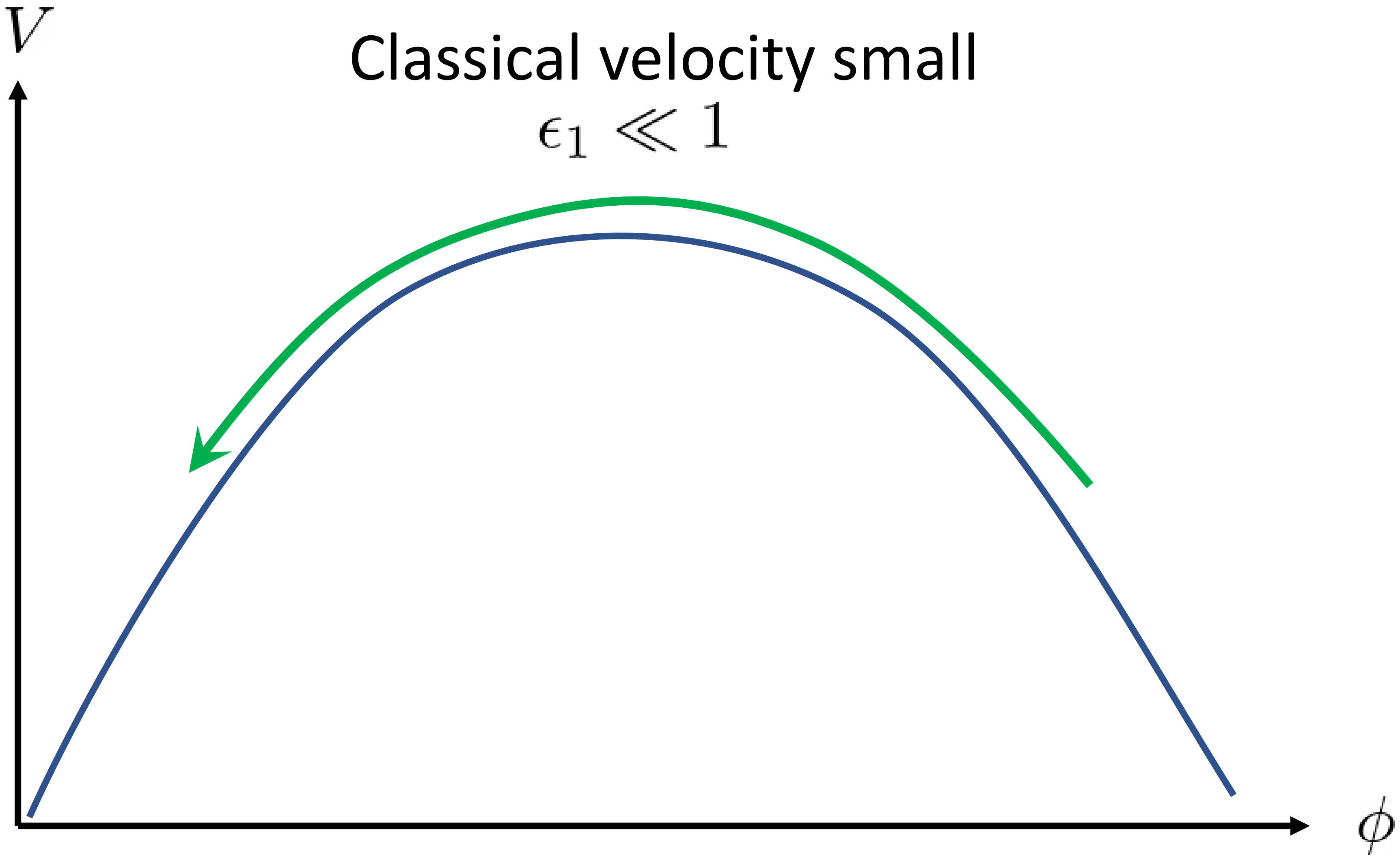
$\langle\xi_\phi(N)\xi_\pi(N$ [illegible] $\frac{}{\mathrm{d}N}\delta\phi_{k_\sigma}(N)\delta\phi'^{*}_{k_\sigma}(N)\delta(N - N')$

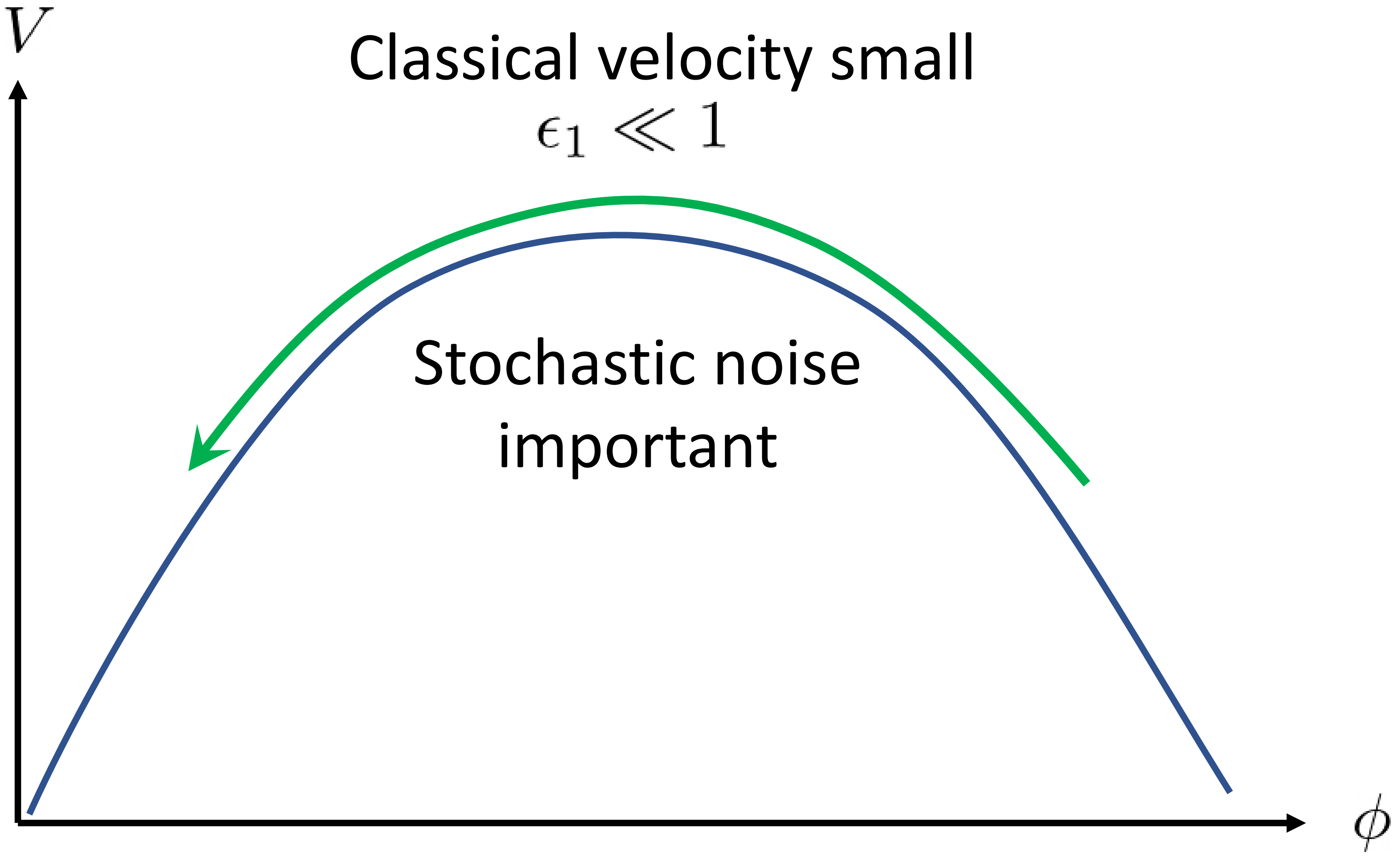
$\mathcal{R} = \Delta N \equiv N - \bar{N}$

COMPLICATED

Classical velocity small

$\epsilon_1 \ll 1$

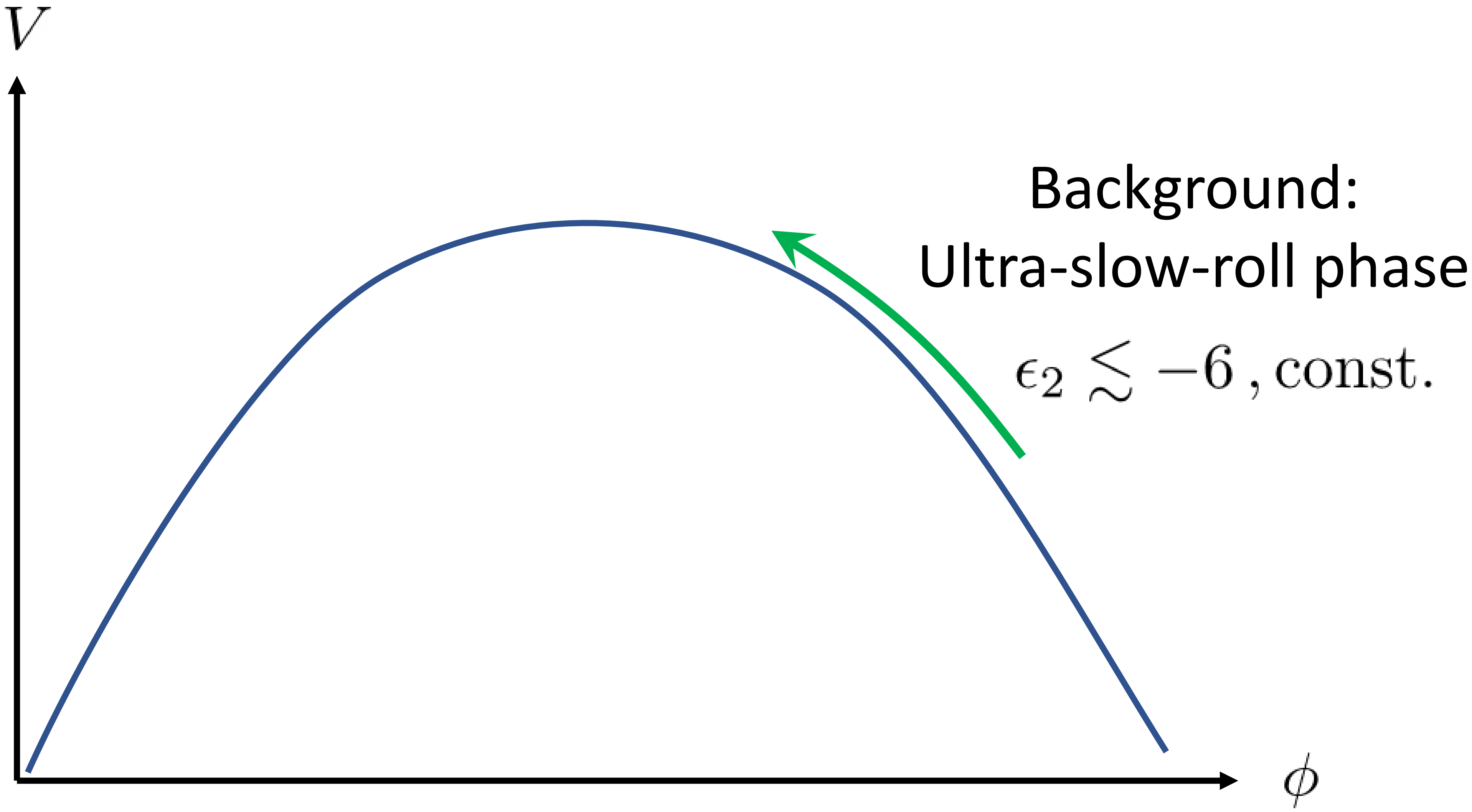




Classical velocity small

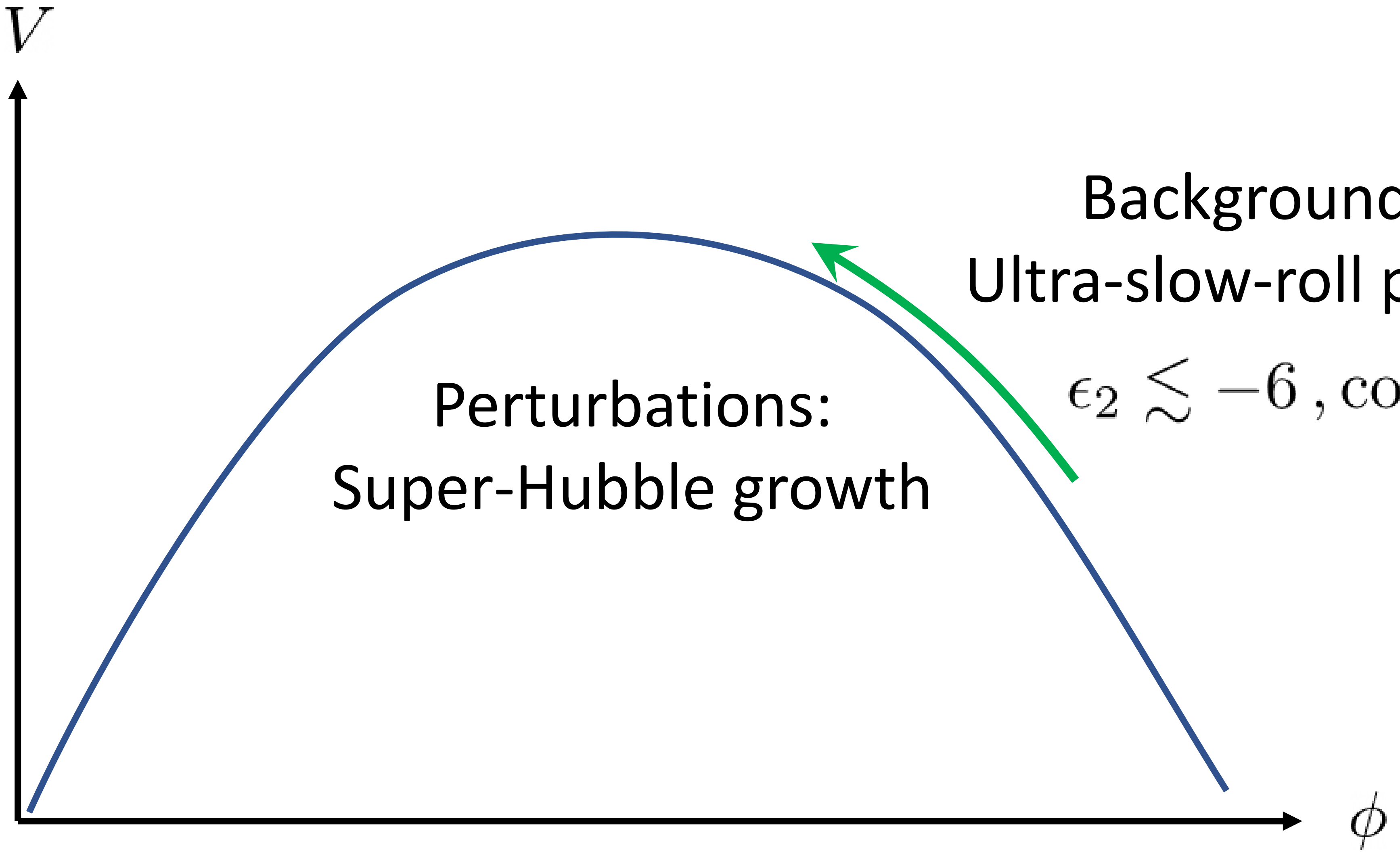
$\epsilon_1 \ll 1$

Stochastic noise important



Background:
Ultra-slow-roll phase

$\epsilon_2 \lesssim -6\,,\mathrm{const.}$



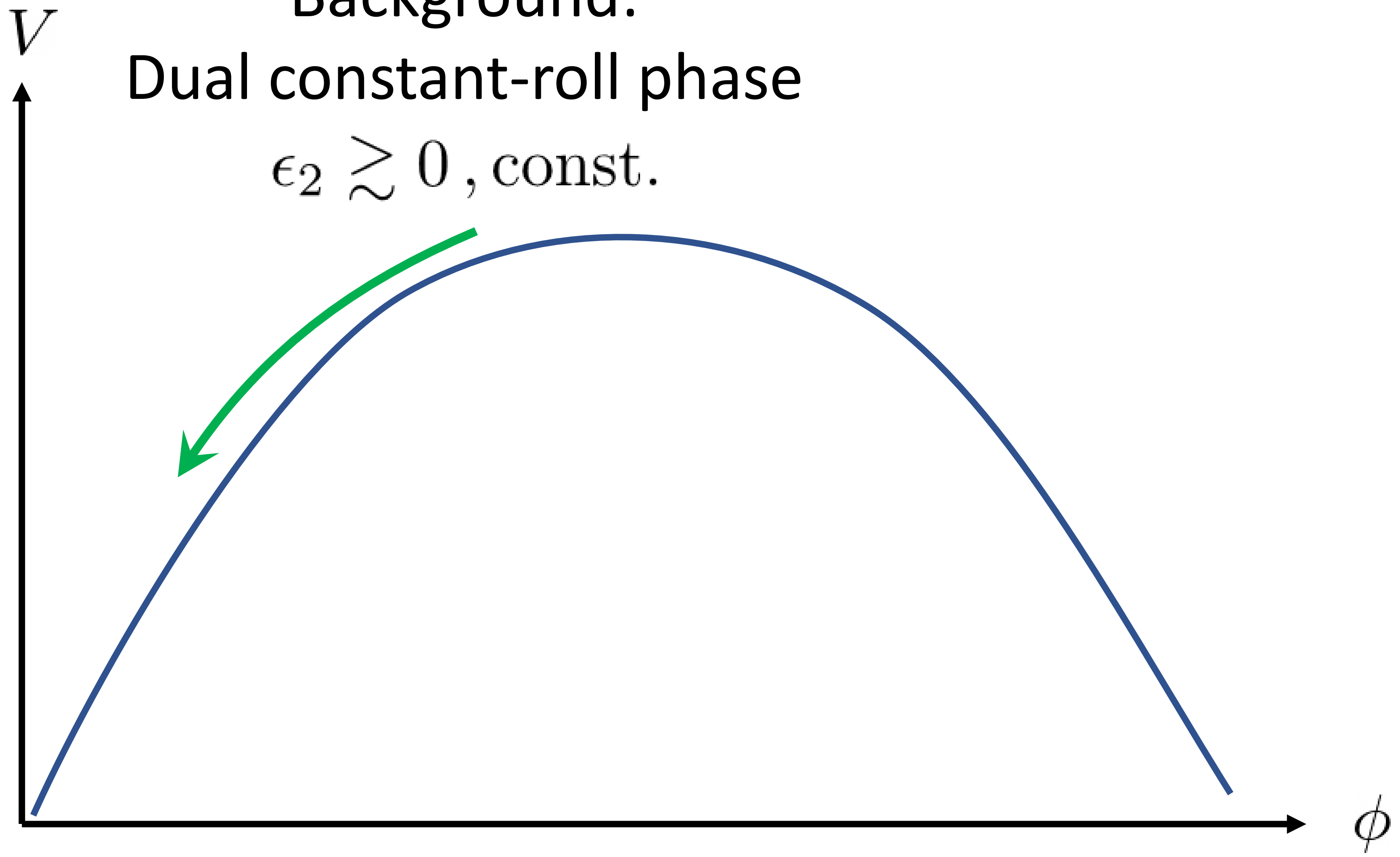
Background:
Ultra-slow-roll phase

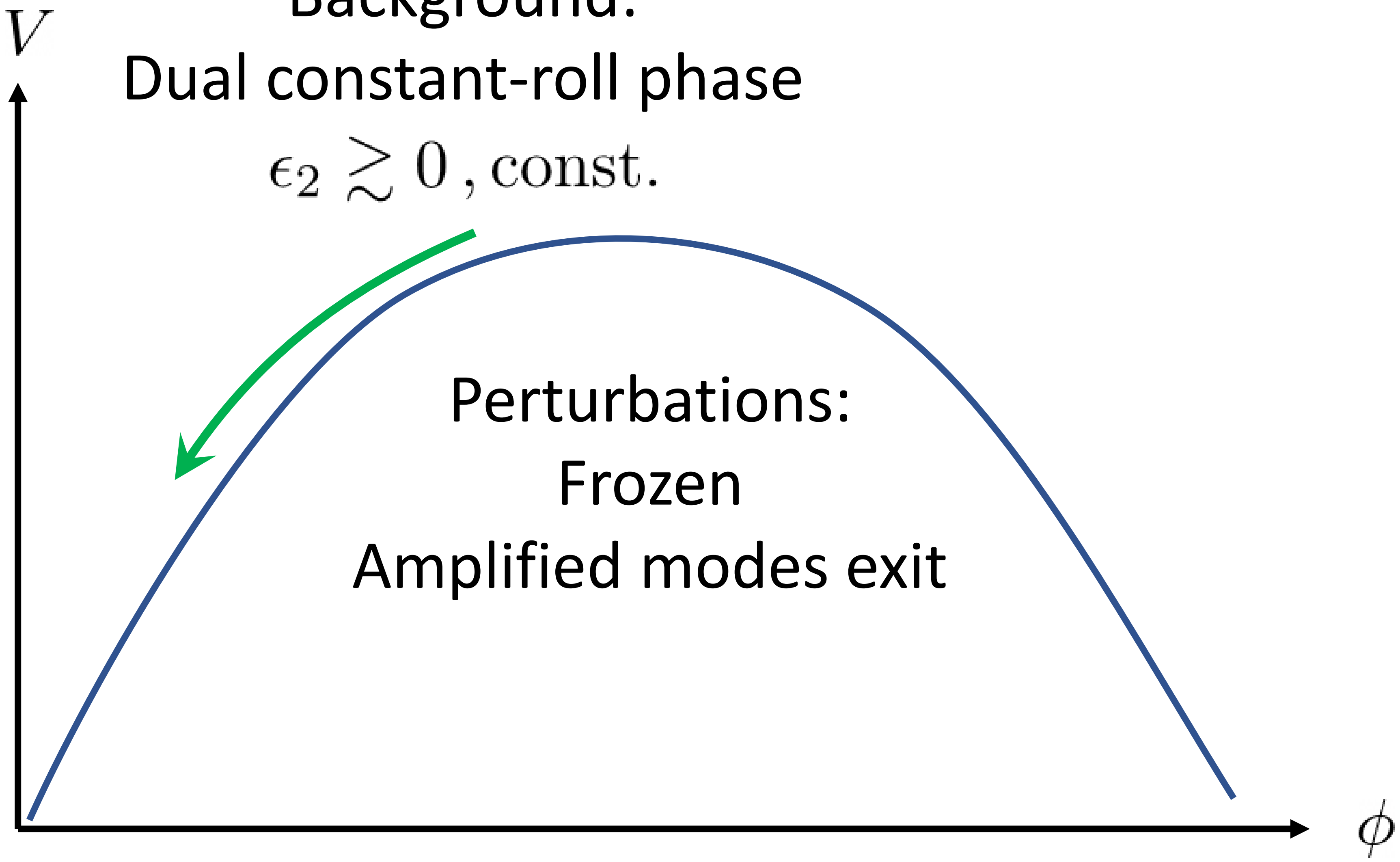
$\epsilon_2 \lesssim -6\,, \mathrm{const.}$

Perturbations:
Super-Hubble growth

Background:
Dual constant-roll phase

$$\epsilon_2 \gtrsim 0\,, \mathrm{const.}$$





Background:
Dual constant-roll phase
$\epsilon_2 \gtrsim 0\,, \mathrm{const.}$

Perturbations:
Frozen
Amplified modes exit

Equations simplify in dual constant-roll phase

Adiabatic perturbations:
motion along classical trajectory only

Noise independent of background stochasticity:
pre-compute power spectrum

Simplified stochastic equation:

$$\mathrm{d}\phi = \frac{\epsilon_2}{2}(\phi - \phi_0)\mathrm{d}N + \frac{\epsilon_2}{2}\phi_0 e^{\frac{\epsilon_2}{2}N}\sqrt{\mathcal{P}_{\mathcal{R}}(k_\sigma)\mathrm{d}N}\,\hat{\xi}_N$$

$$\langle \hat{\xi}_N \hat{\xi}_{N'} \rangle = \delta_{NN'}$$

Simplified stochastic equation:

$$\mathrm{d}\phi = \frac{\epsilon_2}{2}(\phi - \phi_0)\mathrm{d}N + \frac{\epsilon_2}{2}\phi_0 e^{\frac{\epsilon_2}{2}N}\sqrt{\mathcal{P}_{\mathcal{R}}(k_\sigma)\mathrm{d}N}\,\hat{\xi}_N$$

$$\langle \hat{\xi}_N \hat{\xi}_{N'} \rangle = \delta_{NN'}$$

$$\phi(N) = \phi_0\left(1 - e^{\frac{\epsilon_2}{2}N}\right) + \frac{\epsilon_2}{2}\phi_0 e^{\frac{\epsilon_2}{2}N} X(N)$$

$$X(N) \equiv \sum_{k=k_{\mathrm{ini}}}^{k=k_\sigma(N)} \sqrt{\mathcal{P}_{\mathcal{R}}(k)\,\mathrm{d}\ln k}\,\hat{\xi}_k$$

ΔN distribution

$$p(X) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{X^2}{2\sigma^2}} \,, \qquad \sigma^2 \equiv \int_{k_{\mathrm{ini}}}^{k_{\mathrm{end}}} \mathcal{P}_{\mathcal{R}}(k)\, \mathrm{d}\ln k$$

ΔN distribution

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$$X = \frac{2}{\epsilon_2} \left(1 - e^{-\frac{\epsilon_2}{2} \Delta N} \right)$$

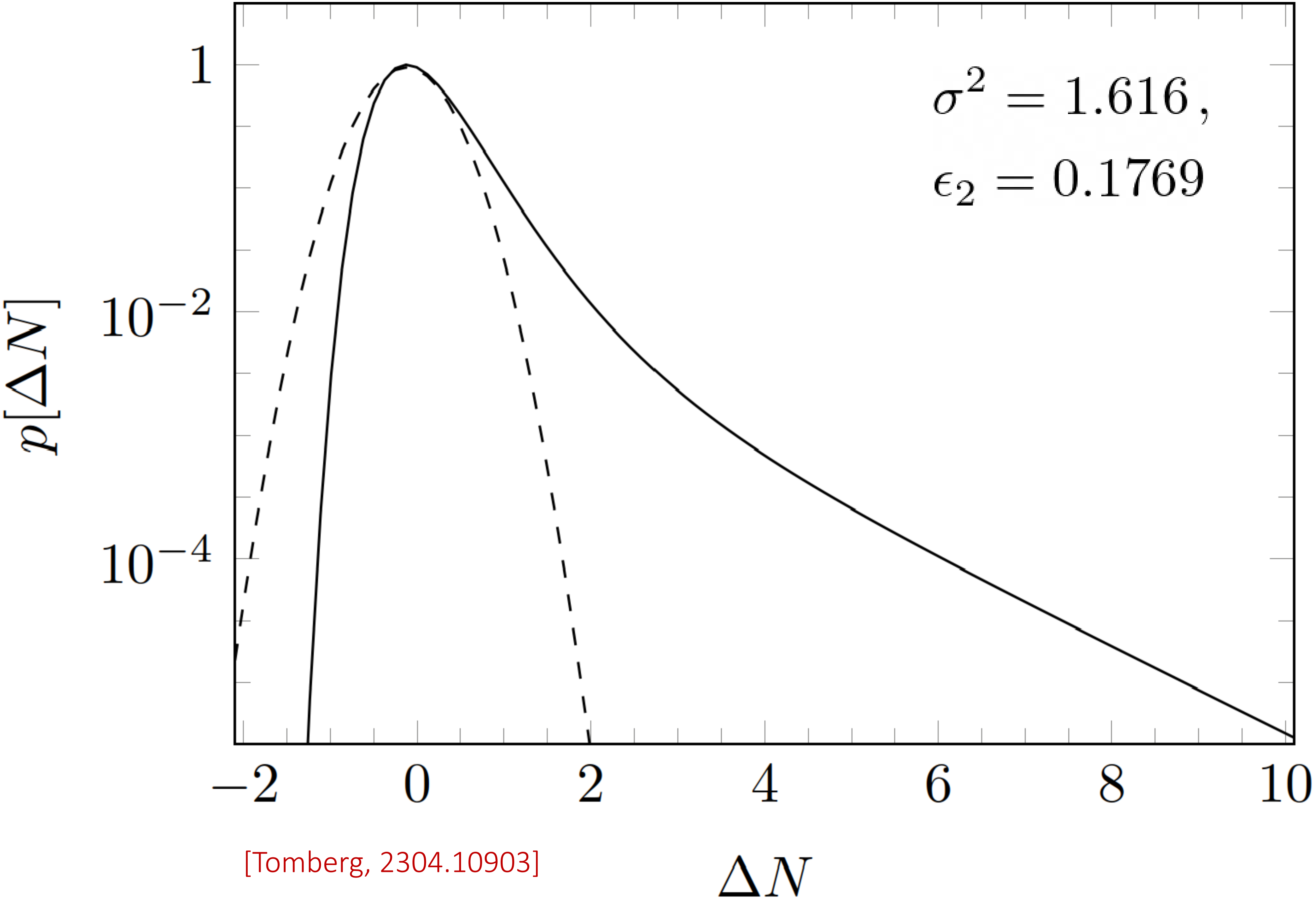
ΔN distribution

$$p(X) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{X^2}{2\sigma^2}}\,, \qquad \sigma^2 \equiv \int_{k_{\mathrm{ini}}}^{k_{\mathrm{end}}} \mathcal{P}_{\mathcal{R}}(k)\,\mathrm{d}\ln k$$

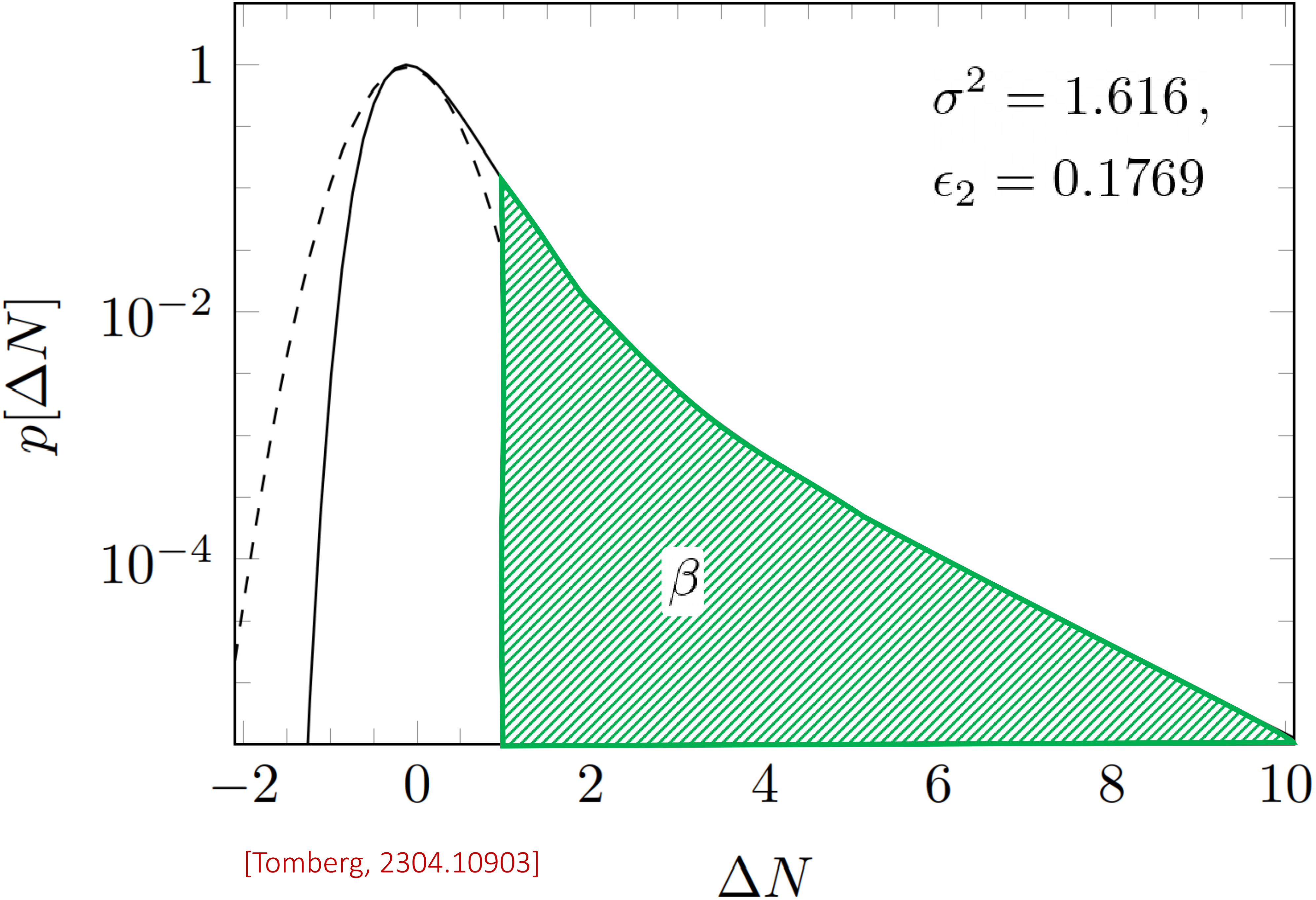
$$X = \frac{2}{\epsilon_2}\left(1 - e^{-\frac{\epsilon_2}{2}\Delta N}\right)$$

$$\boxed{p(\Delta N) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{2}{\sigma^2\epsilon_2^2}\left(1 - e^{-\frac{\epsilon_2}{2}\Delta N}\right)^2 - \frac{\epsilon_2}{2}\Delta N\right]}$$

$\Delta N = \mathcal{R}$



[Tomberg, 2304.10903]



[Tomberg, 2304.10903]

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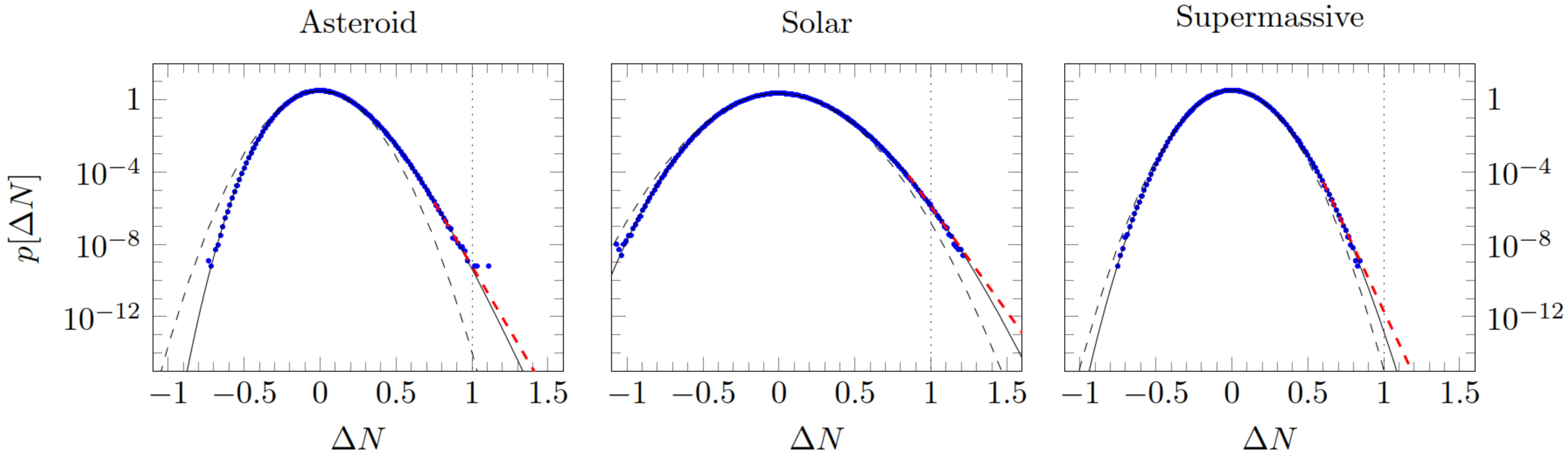


Black = Constant-roll approximation

Dashed = Gaussian fit

Blue = numerical computation

Red = numerical extrapolation

Sneak peek: compaction function

$$\mathcal{C}(r) = \frac{2}{3}\left(1 - [1 + r\zeta'(r)]^2\right)$$

$$r\zeta'(r) = \sum_k \left[- \frac{\hat{\xi}_k}{1 - \frac{\epsilon_2}{2} X(k)} \sqrt{\mathcal{P}_\zeta(k)\,\mathrm{d}\ln k} + \frac{\epsilon_2}{4\left[1 - \frac{\epsilon_2}{2} X(k)\right]^2} \mathcal{P}_\zeta(k)\,\mathrm{d}\ln k \right]$$

$$\times \left[\cos(kr) - \frac{\sin(kr)}{kr} \right]$$

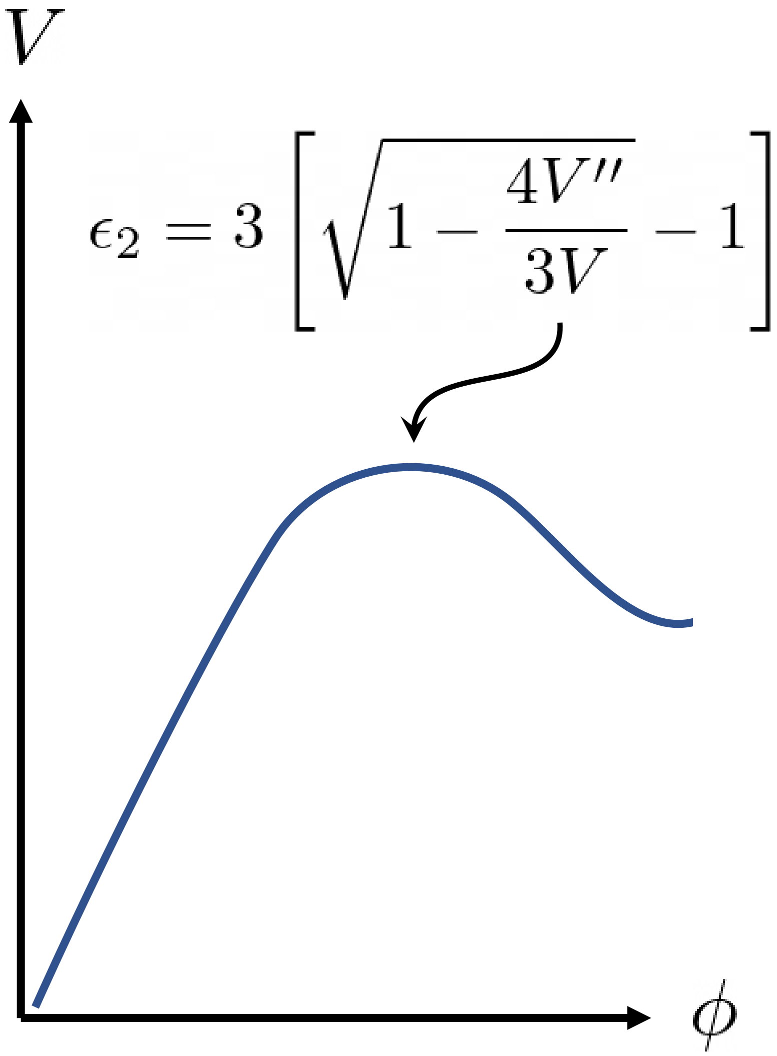
Conclusions

Stochastic inflation introduces non-Gaussian corrections to PBH statistics

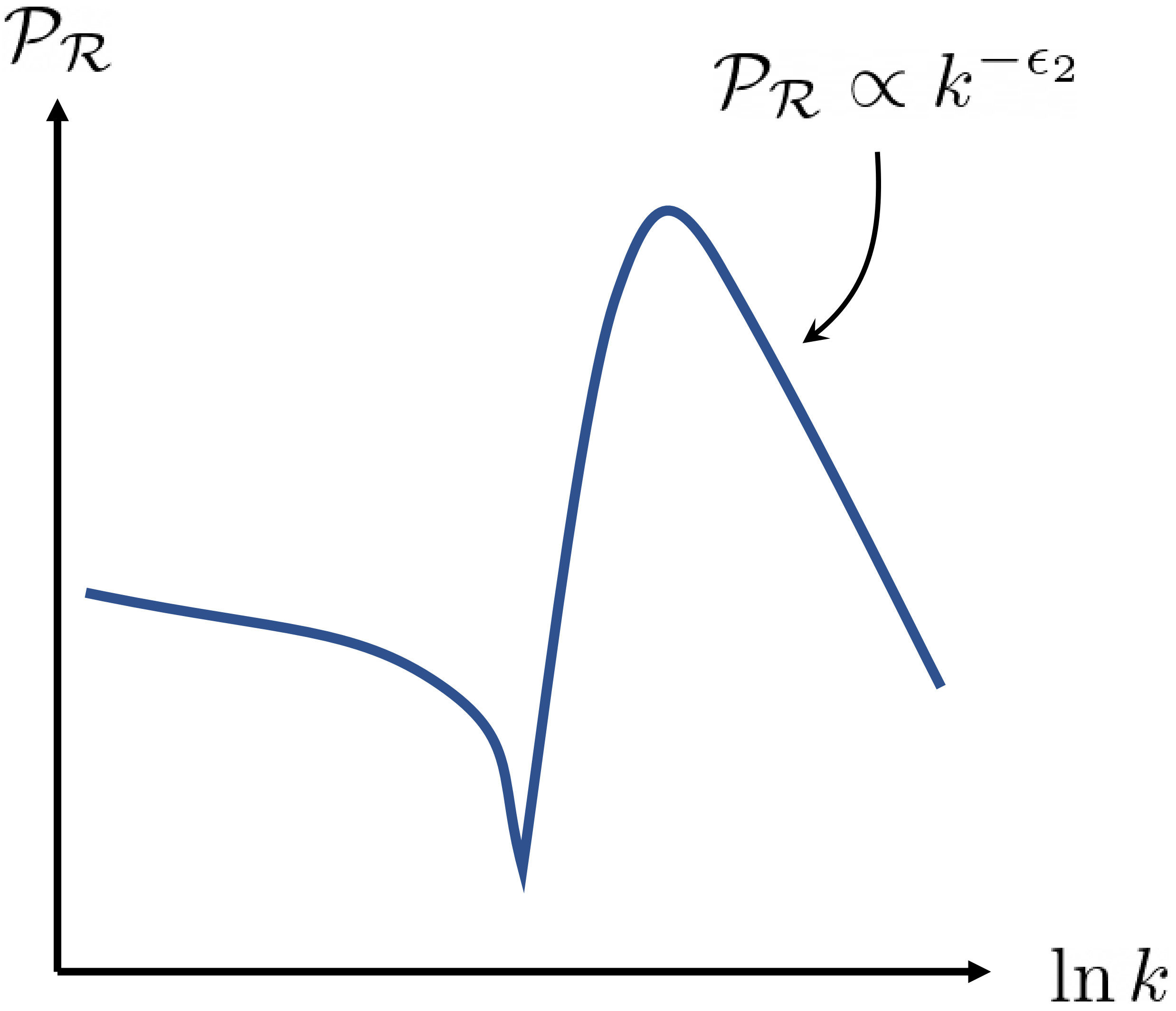
Constant-roll inflation is at the heart of PBH inflation models

$$\boxed{p(\mathcal{R}) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{2}{\sigma^2\epsilon_2^2}\left(1 - e^{-\frac{\epsilon_2}{2}\mathcal{R}}\right)^2 - \frac{\epsilon_2}{2}\mathcal{R}\right]}$$

[Karam et al, 2205.13540]



$$\epsilon_2 = 3\left[\sqrt{1-\frac{4V''}{3V}}-1\right]$$



$$\mathcal{P}_\mathcal{R} \propto k^{-\epsilon_2}$$