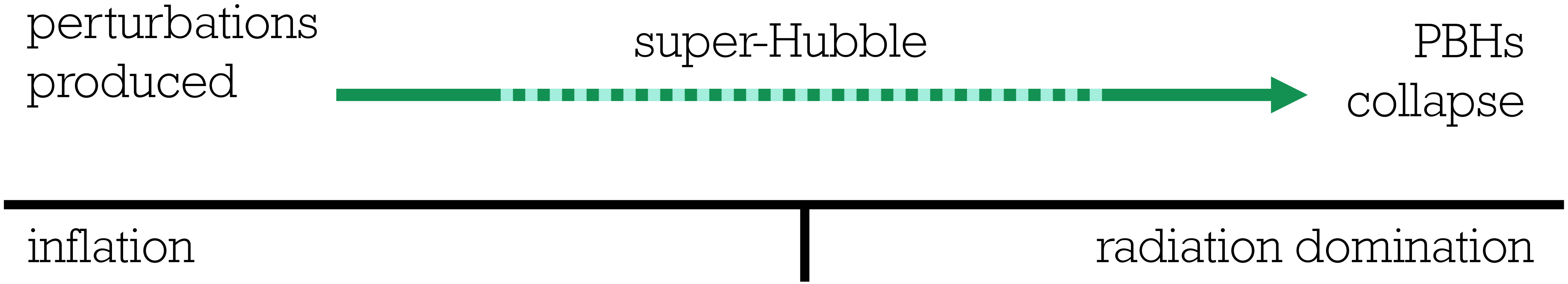
Compaction function from stochastic inflation (and other topics)

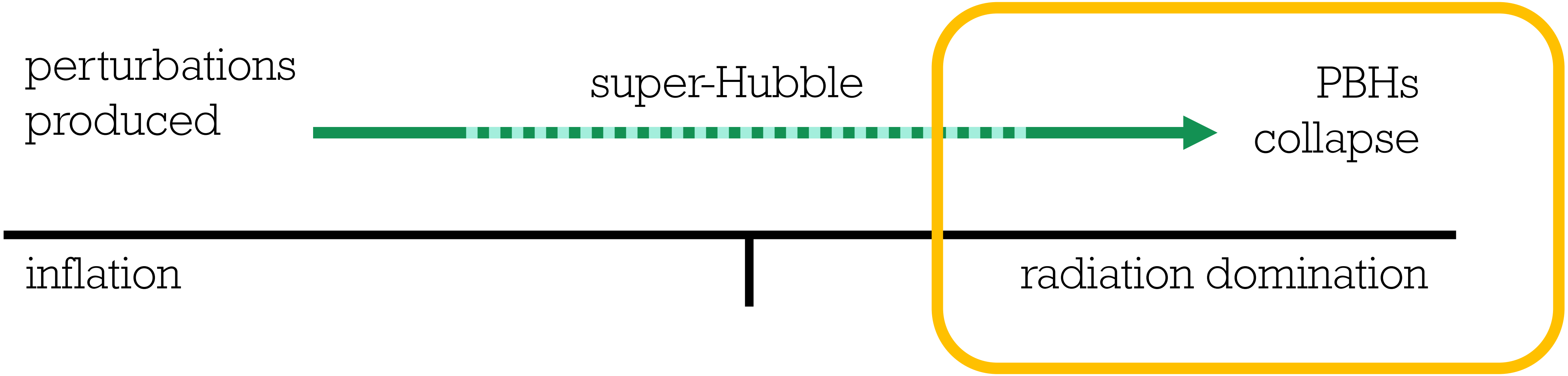
IFPU, TRIESTE, 3 NOVEMBER 2025

EEMELI TOMBERG

PBHs from inflation



PBHs from inflation



Which perturbations collapse?

Non-linear process

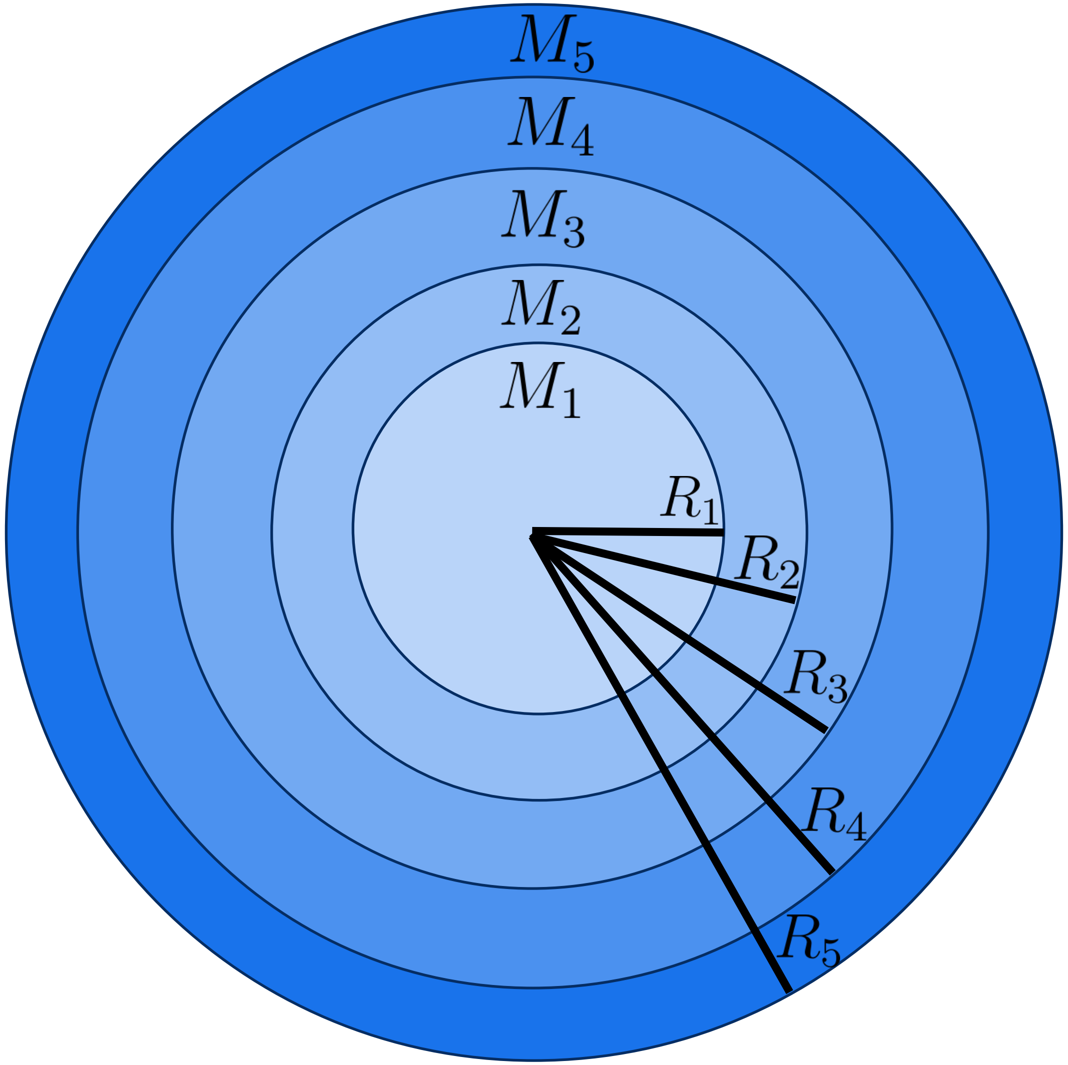
Initial conditions:
δ, metric perturbations

Which perturbations collapse?

Numerical relativity

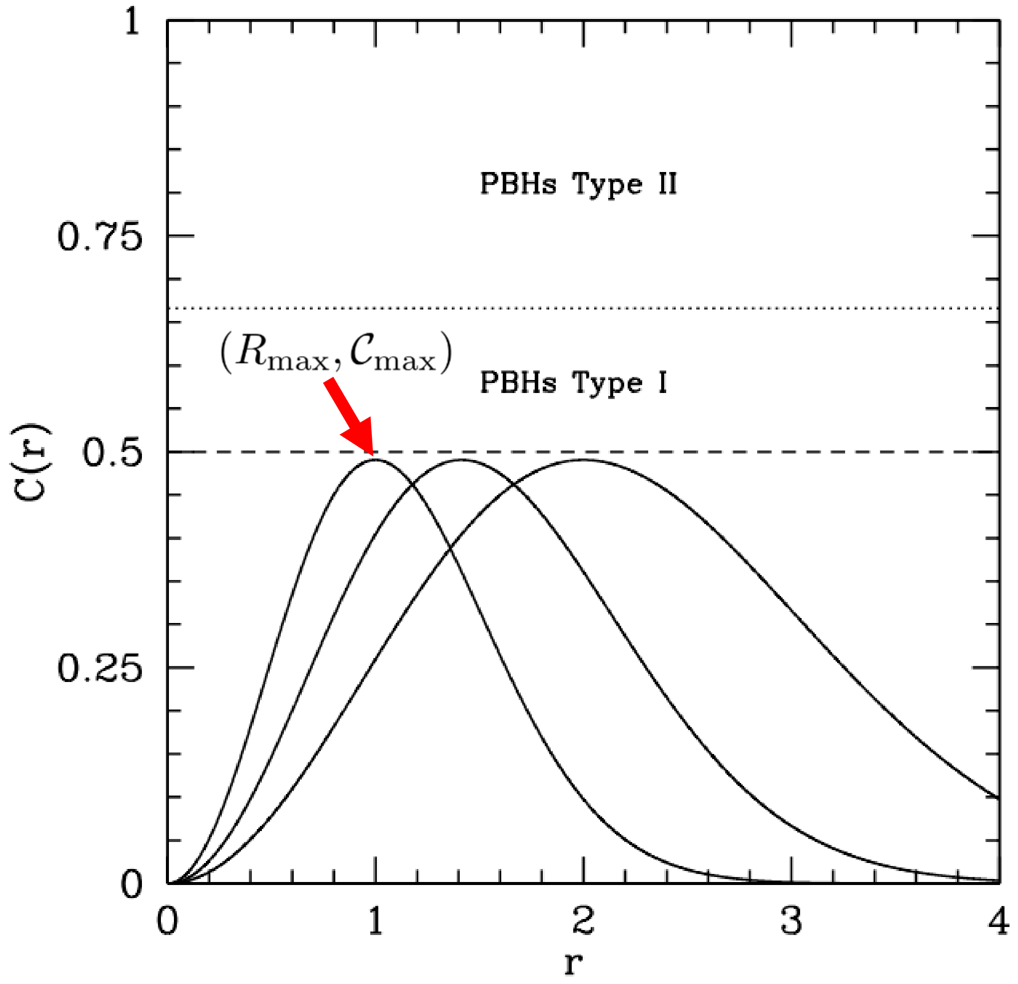
Compaction function:

$$\mathcal{C}(r,t) = 2\frac{\delta M(r,t)}{R(r,t)}$$



Collapse simulations

Musco:2018rwt

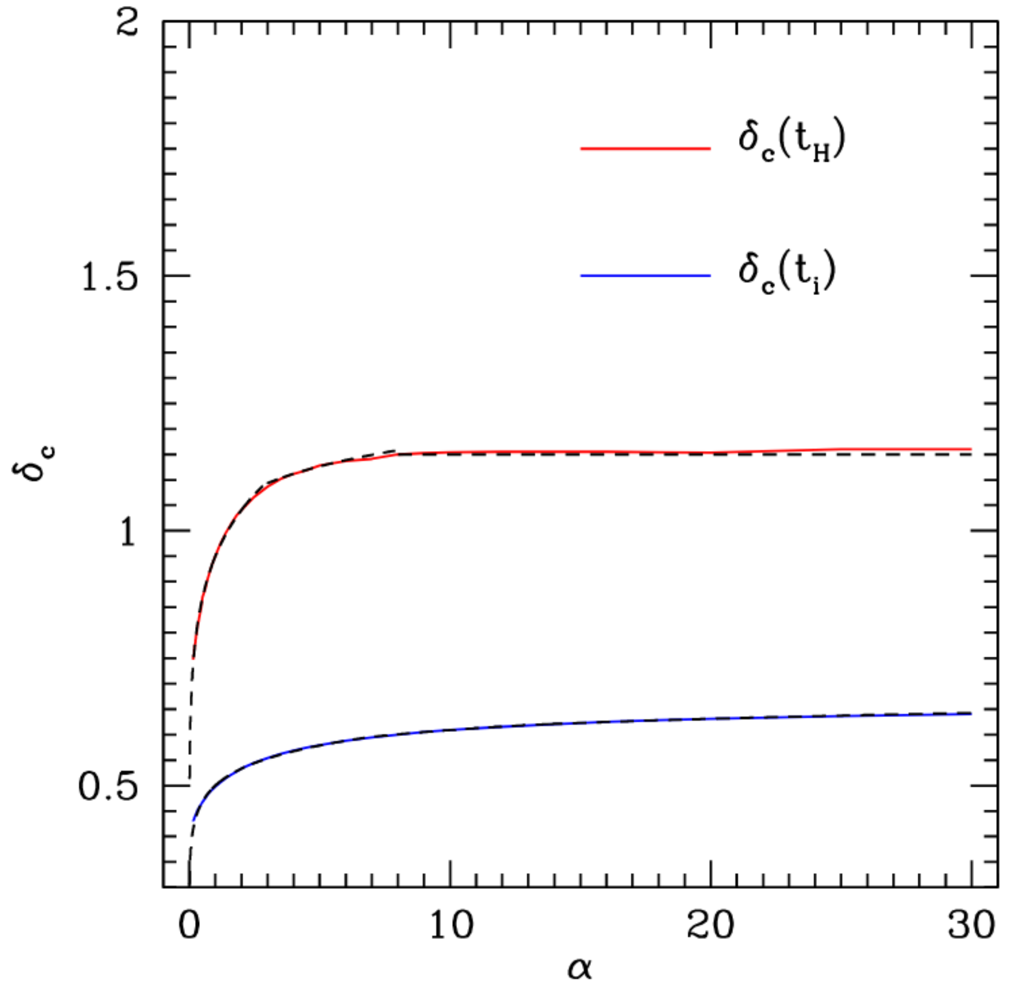


Collapse:

$$\mathcal{C}_{\max} > \mathcal{C}_{\rm th} = \delta_c$$

Collapse simulations

Musco:2020jjb



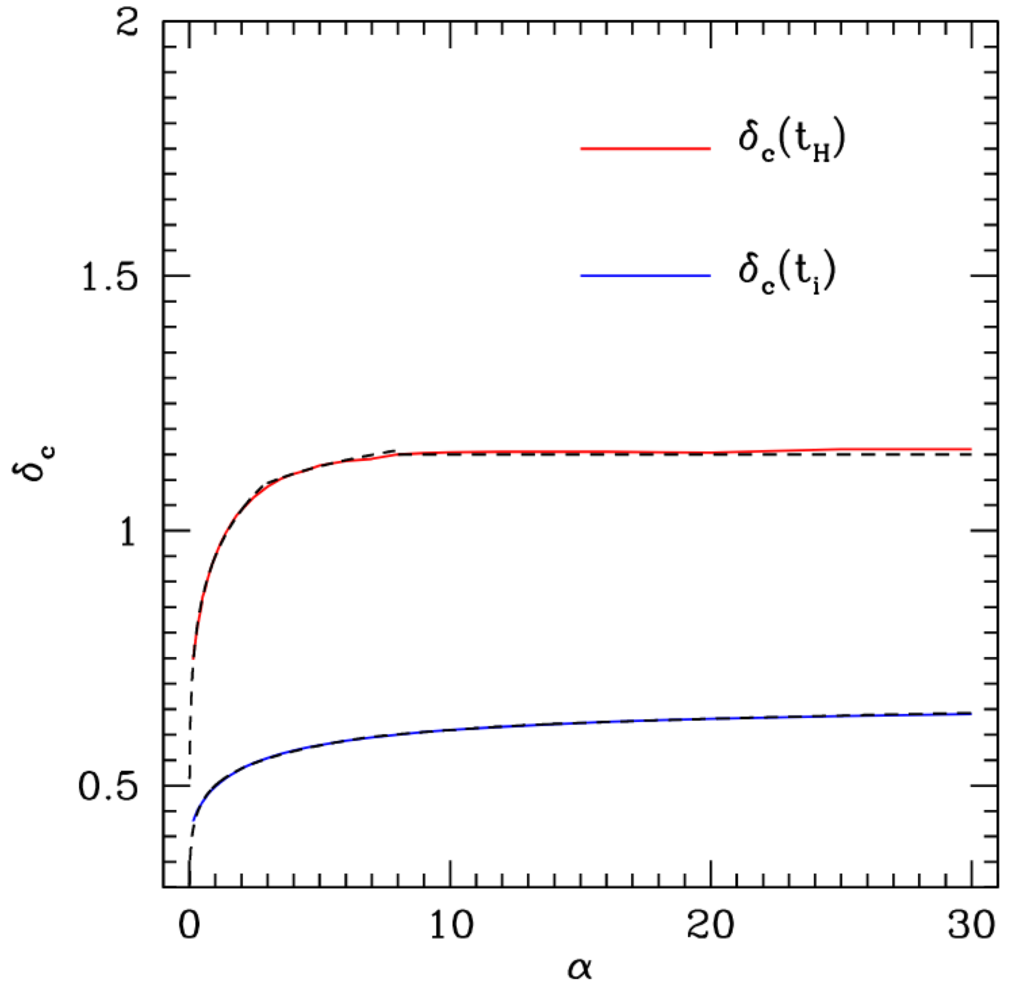
Collapse:

$$\mathcal{C}_{\rm max} > \mathcal{C}_{\rm th} = \delta_c$$

$$\alpha = -\frac{\mathcal{C}''(R_{\rm max})R_{\rm max}^2}{4\mathcal{C}_{\rm max}}$$

Collapse simulations

Musco:2020jjb



Collapse:

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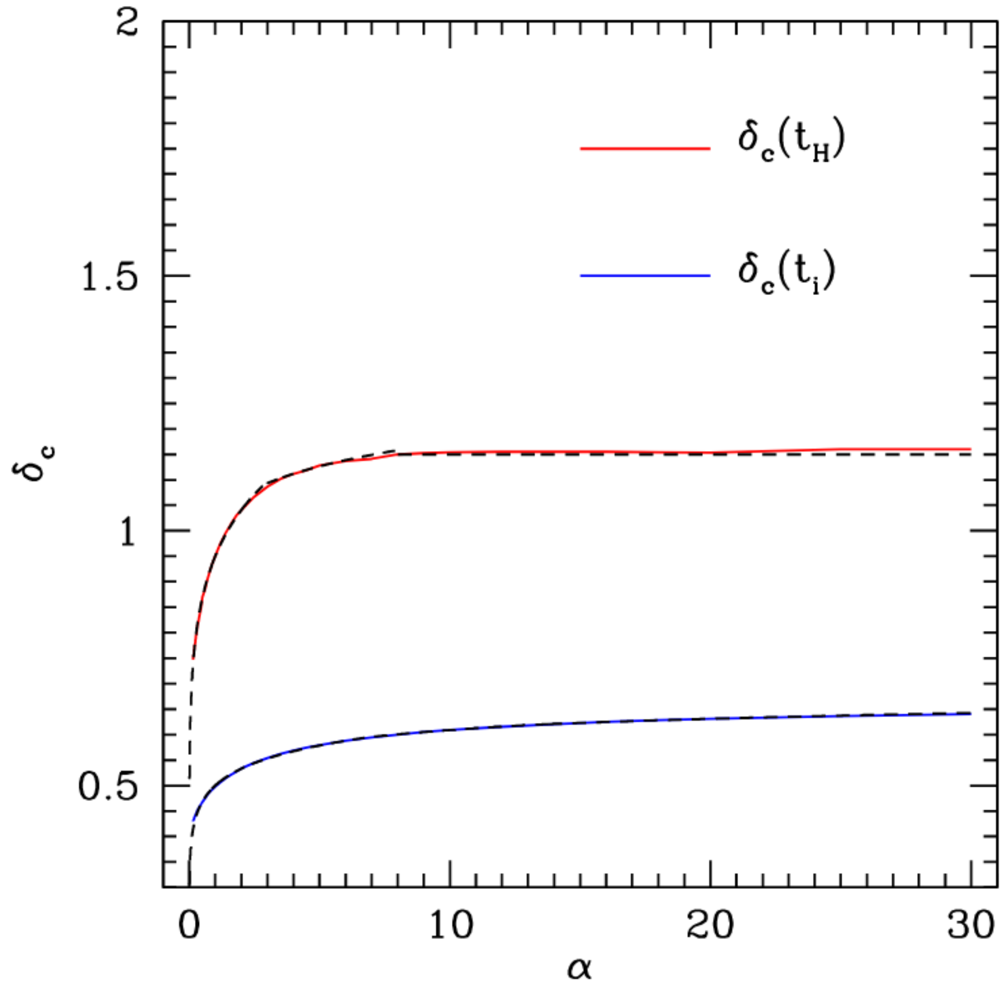
$$\alpha = -\frac{\mathcal{C}''(R_{\max})R_{\max}^2}{4\mathcal{C}_{\max}}$$

Escriva:2019phb

$$\bar{\mathcal{C}} = \frac{3}{R_{\max}^3}\int_0^{R_{\max}} \mathcal{C}(r)R^2 \mathrm{d}R > 0.4$$

Collapse simulations

Musco:2020jjb



Collapse:

$$\mathcal{C}_{\rm max} > \mathcal{C}_{\rm th} = \delta_c$$

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Escriva:2019phb

$$\bar{\mathcal{C}} = \frac{3}{R_{\rm max}^3}\int_0^{R_{\rm max}} \mathcal{C}(r) R^2 {\rm d}R > 0.4$$

PBH mass from $R_{\rm max}$

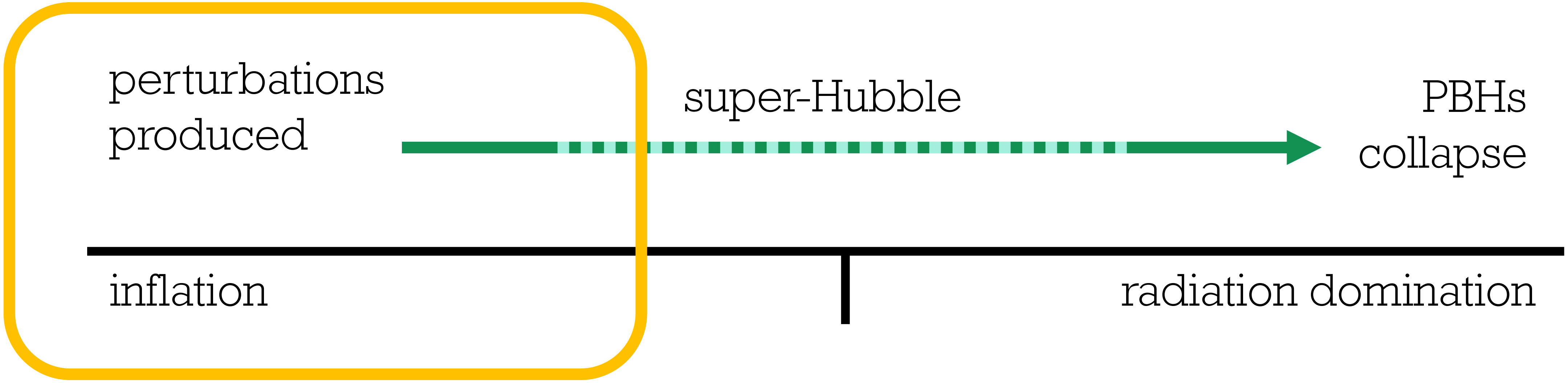
Compaction versus curvature

Cosmological perturbation theory:
curvature perturbation ζ

Super-Hubble:

$$\mathcal{C}(r) = \frac{2}{3}\left(1 - [1 + r\zeta'(r)]^2\right)$$

PBHs from inflation



Perturbations from inflation

Linear perturbation theory

Curvature perturbation ζ

Single-field inflation: background

Equations of motion:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0\,, \qquad 3H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

Slow-roll parameters:

$$\epsilon_1 \equiv -\partial_N \ln H\,, \qquad \epsilon_2 \equiv \partial_N \ln \epsilon_1$$

Inflation?

Slow-roll?

Single-field inflation: perturbations

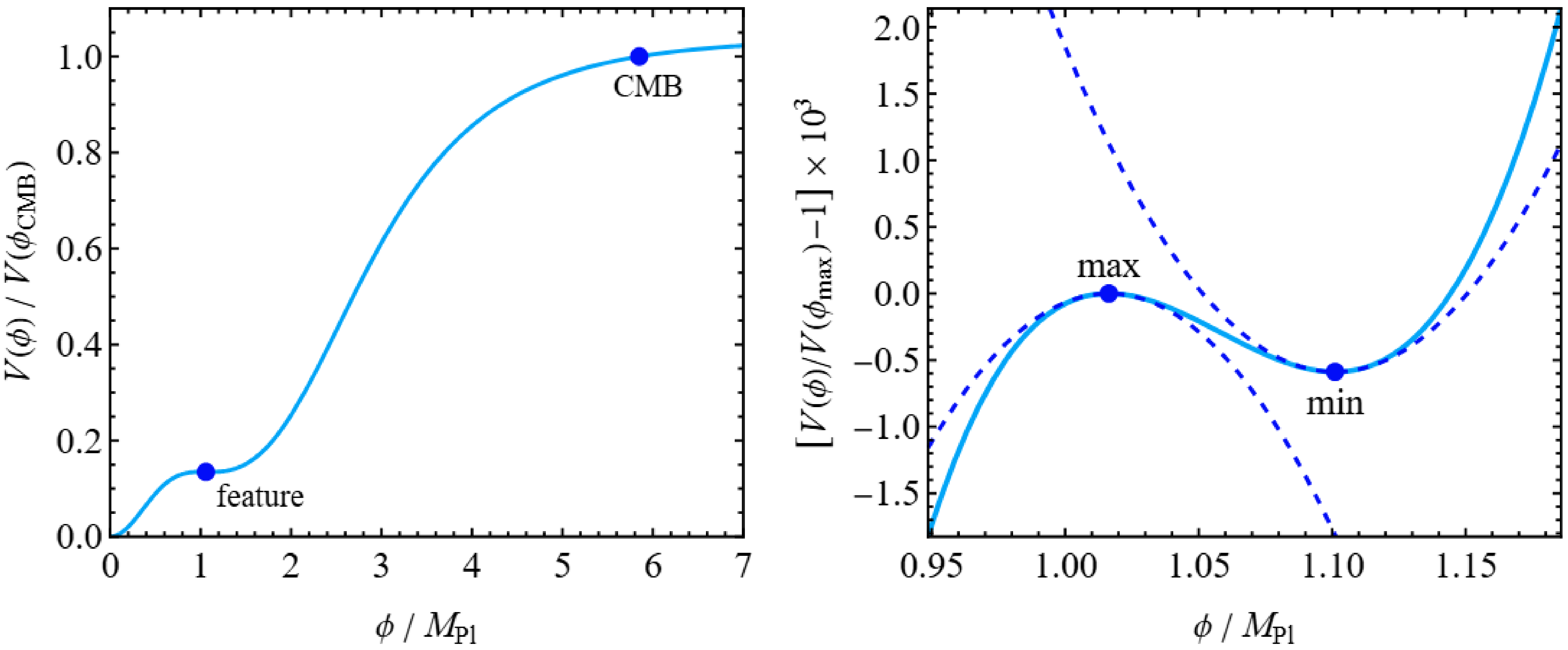
Field perturbations:

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} + \left[\frac{k^2}{a^2} + V''(\phi) - \frac{1}{a^3}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a^3}{H}\dot{\phi}^2\right)\right]\delta\phi = 0$$

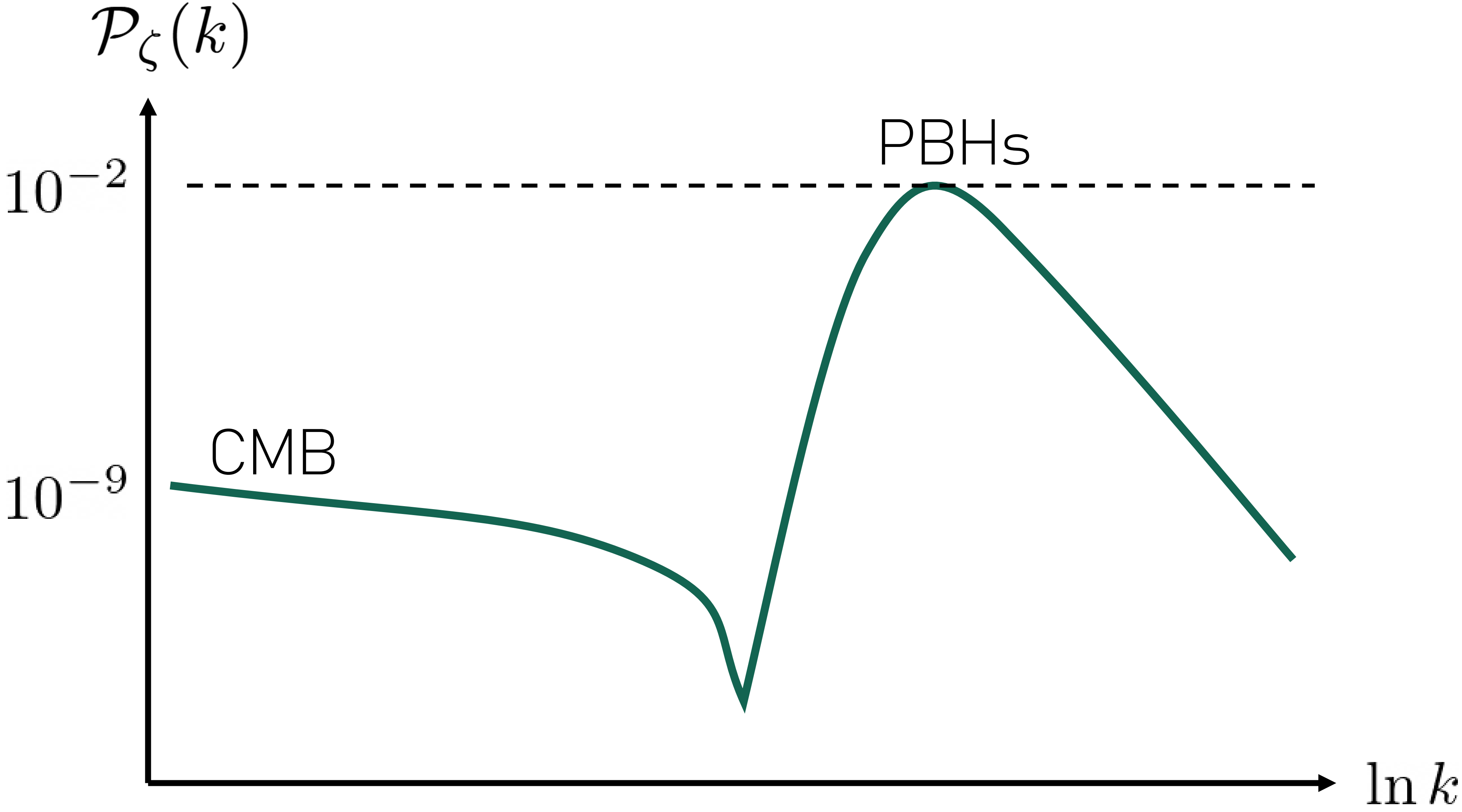
Curvature perturbation:

$$\zeta_k = \frac{\delta\phi_k}{\sqrt{2\epsilon_1}} \quad \leftarrow \sim H \qquad\qquad \mathcal{P}_\zeta(k) = \frac{k^3}{2\pi^2}|\zeta_k|^2$$

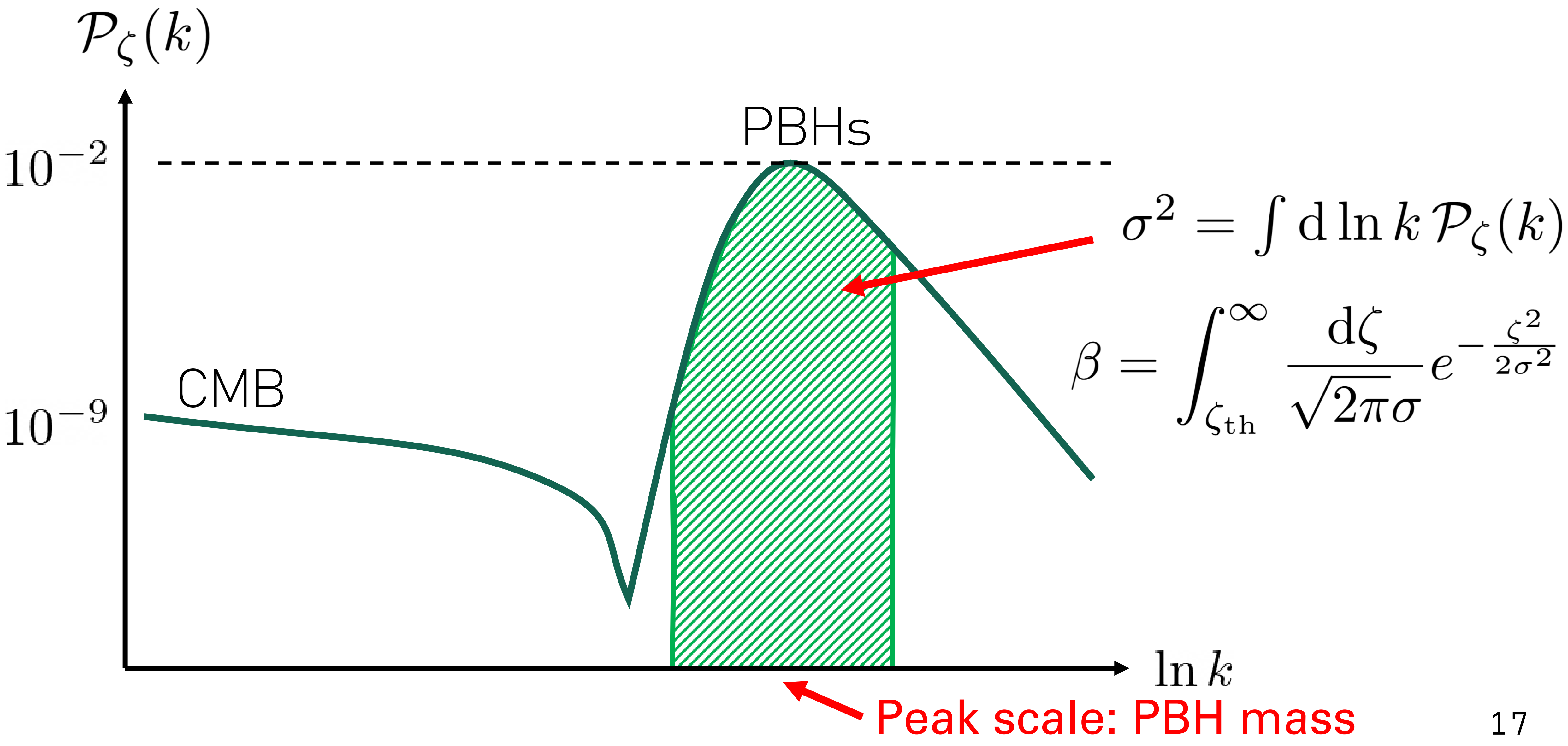
Inflection point models



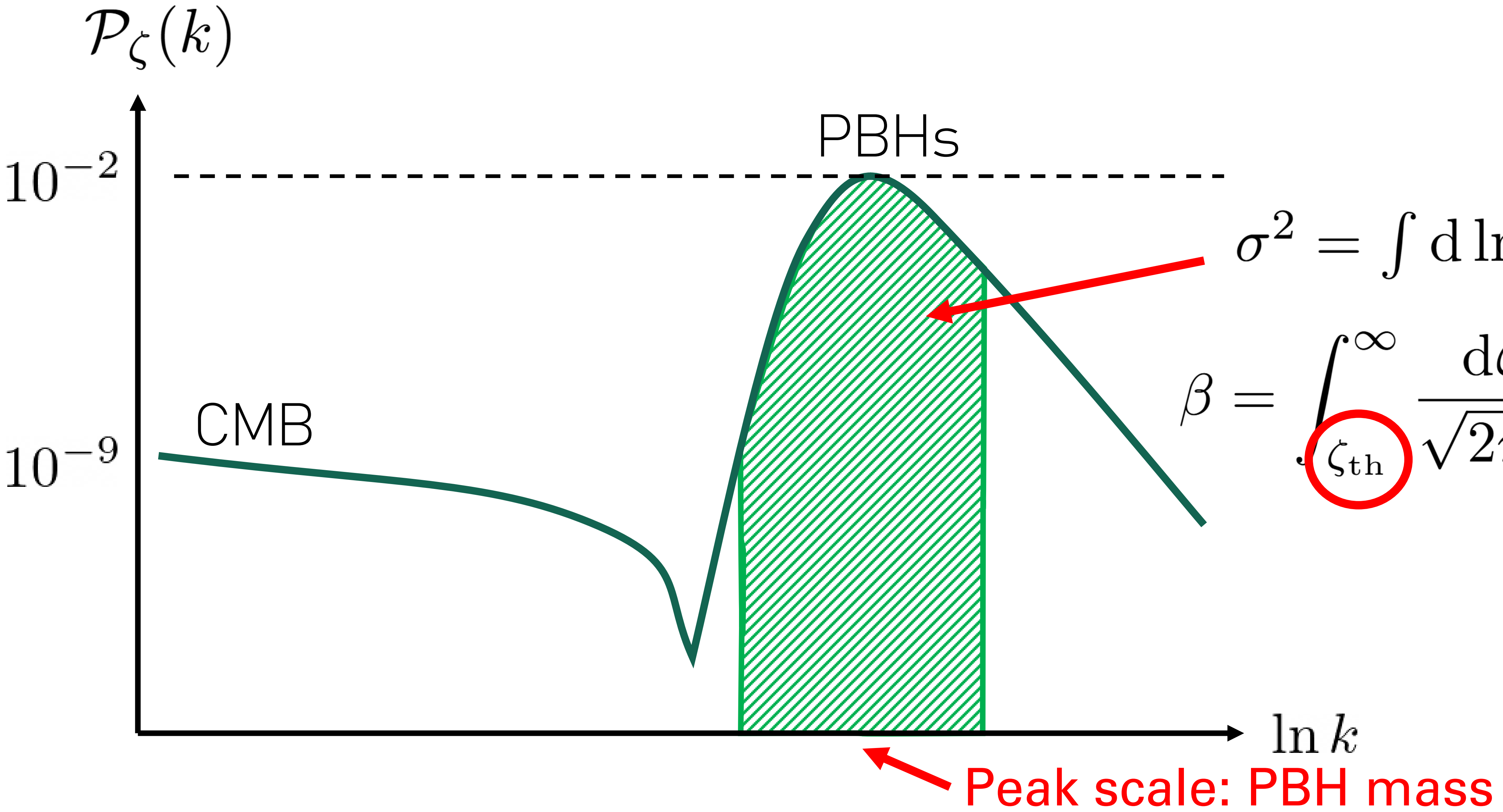
Inflection point models



Inflection point models



Inflection point models



$$\sigma^2 = \int \mathrm{d}\ln k\, \mathcal{P}_\zeta(k)$$

$$\beta = \int_{\zeta_{\rm th}}^{\infty} \frac{\mathrm{d}\zeta}{\sqrt{2\pi}\sigma} e^{-\frac{\zeta^2}{2\sigma^2}}$$

Compaction function?

Gaussian perturbations: mean profile

Gaussian

$$\zeta(r) = \frac{\zeta_0}{\sigma^2} \int \mathrm{d}\ln k\, \mathcal{P}_\zeta(k) \frac{\sin kr}{kr}$$

Then: $\mathcal{C}(r) = \frac{2}{3}\left(1 - [1 + r\zeta'(r)]^2\right)$

$\mathcal{C}$ threshold gives ζ threshold

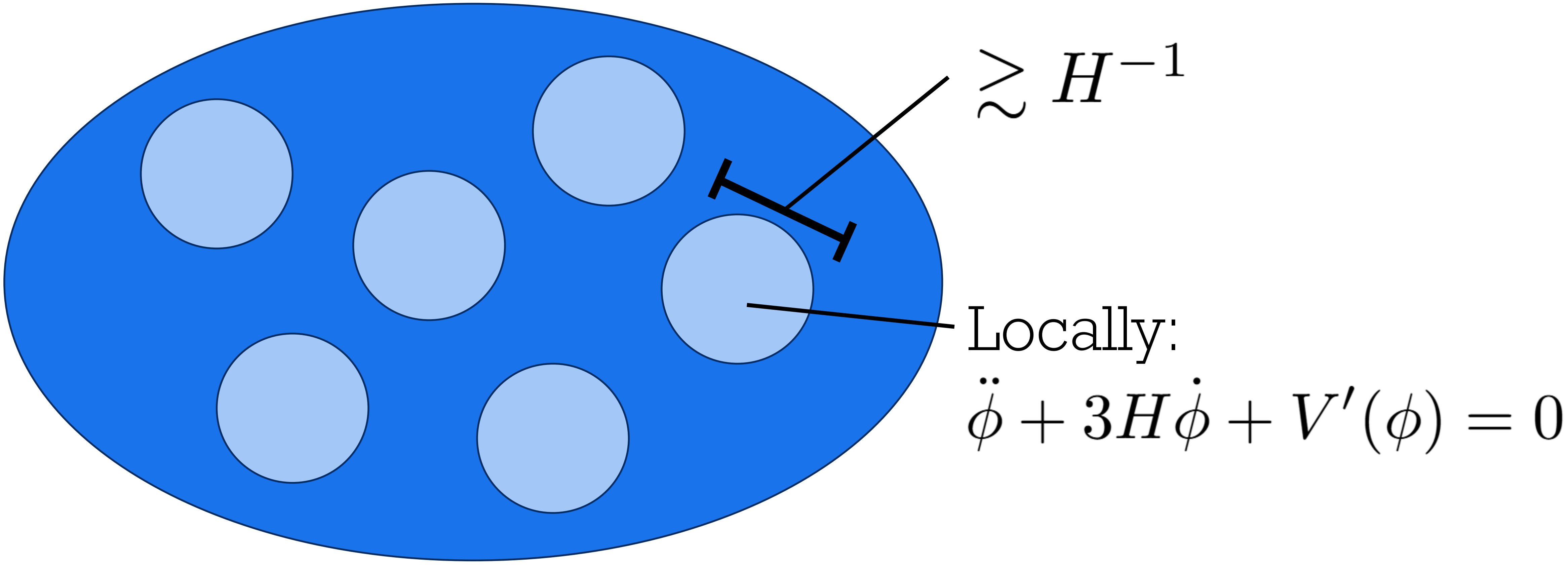
PROBLEMS

Linear perturbations
not enough?

Mean profiles not enough?

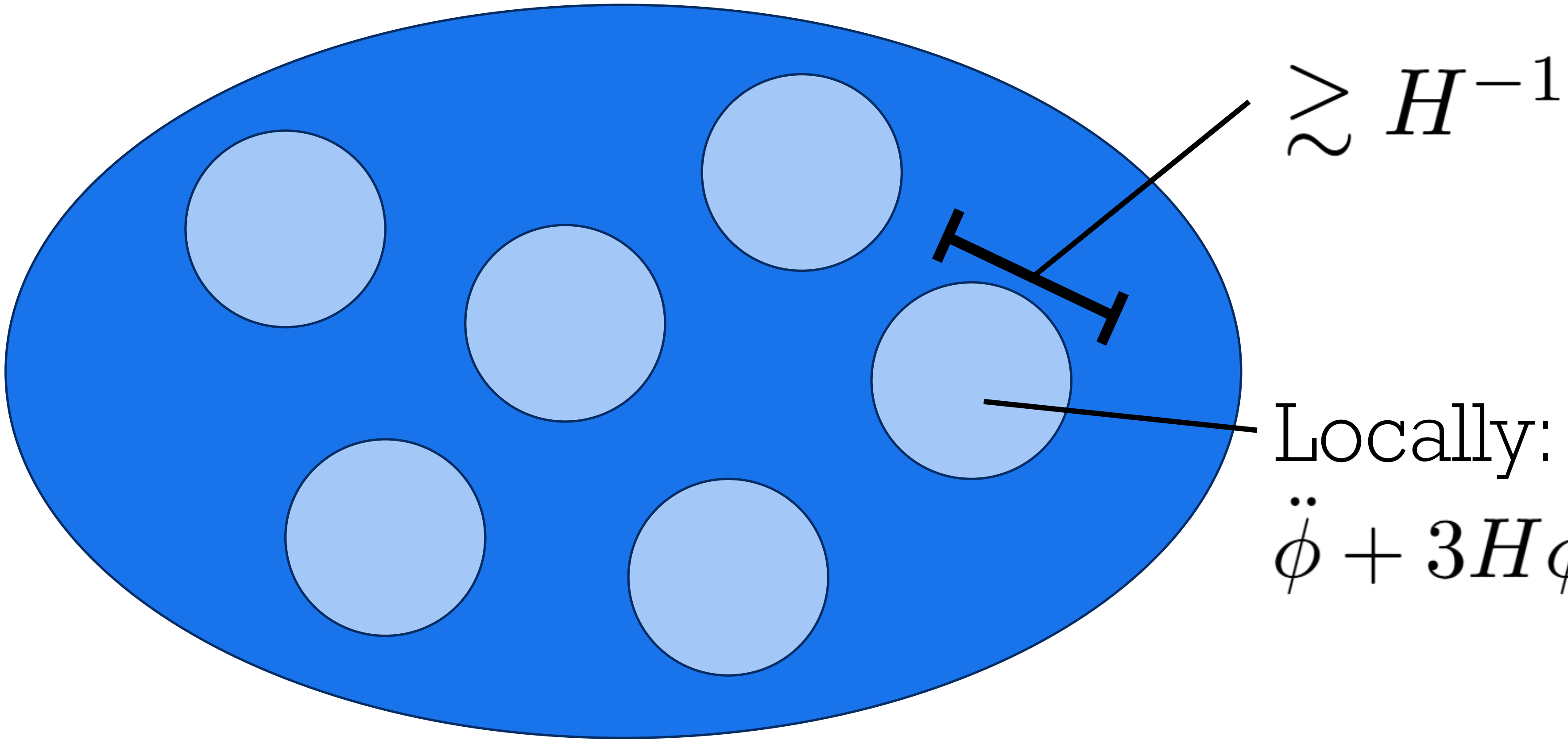
Beyond linear perturbations

Separate universe approximation:



Beyond linear perturbations

Separate universe approximation:



$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

ΔN approximation: $\Delta N = N - \langle N \rangle = \zeta$

$$\ddot{\phi} - \frac{1}{a^2}\nabla^2\phi + 3H\dot{\phi} + V'(\phi) = 0$$

Linear perturbations:
small-$\delta\phi$ expansion

$$\ddot{\phi} - \frac{1}{a^2}\nabla^2\phi + 3H\dot{\phi} + V'(\phi) = 0$$

$$\delta\ddot{\phi} \qquad \delta\phi \qquad \delta\dot{\phi} \qquad V''(\phi)\delta\phi$$

+ metric perts

Separate universe:

small-$\frac{\nabla^2}{a^2H^2}$ expansion

$$\ddot{\phi} - \frac{1}{a^2}\nabla^2\phi + 3H\dot{\phi} + V'(\phi) = 0$$

Divide the field in two

$$\phi_{\rm tot} = \phi + \delta\phi$$

$$\int_{k>k_\sigma} \frac{\mathrm{d}k^3}{(2\pi)^{2/3}} \phi_k(t) e^{-i\vec{k}\cdot\vec{x}}$$

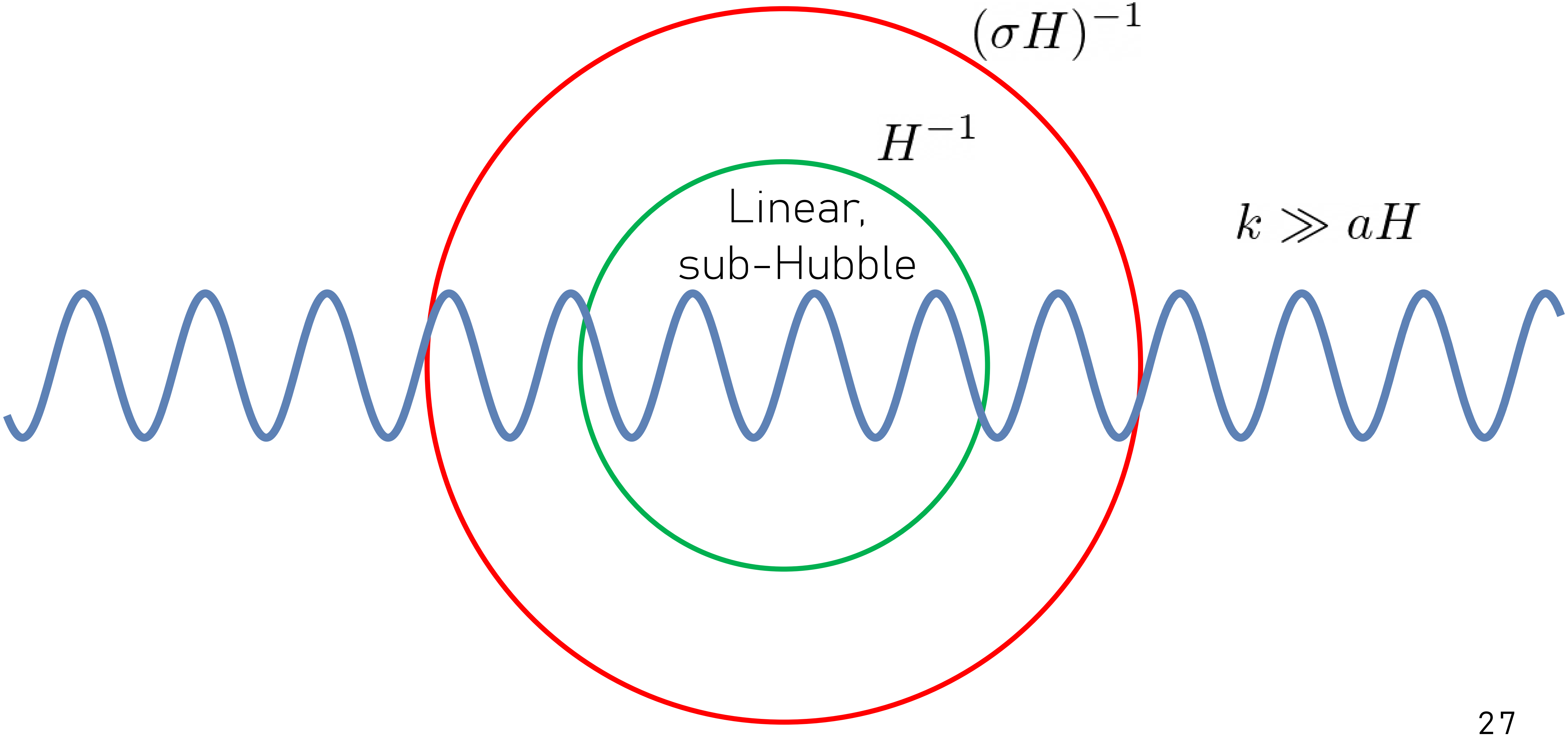
Separate universe

$$\int_{k>k_\sigma} \frac{\mathrm{d}k^3}{(2\pi)^{2/3}} \phi_k(t) e^{-i\vec{k}\cdot\vec{x}}$$

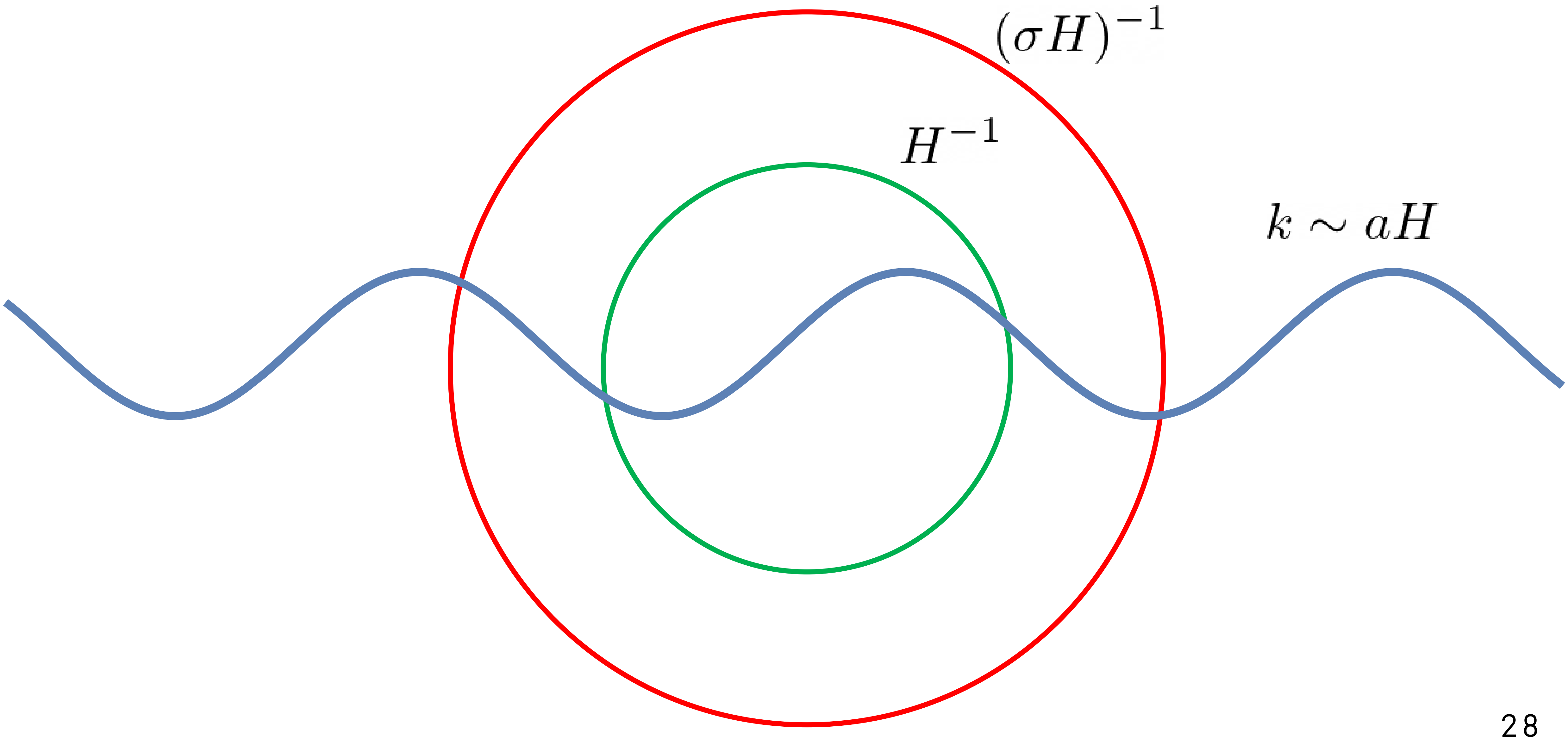
Linear perturbations

Coarse-graining scale: $k = k_\sigma \equiv \sigma a H$

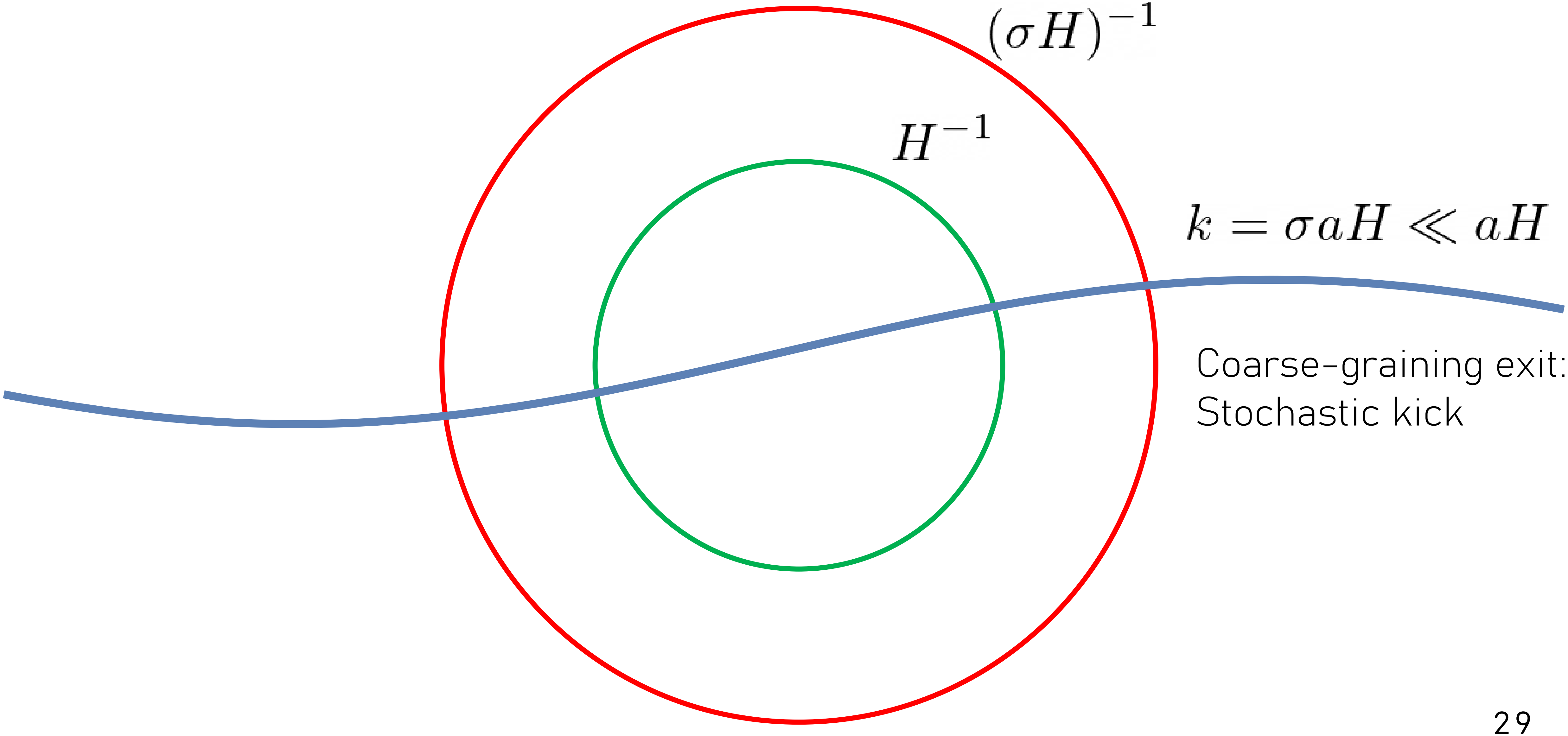
Drifting modes: stochasticity



Drifting modes: stochasticity



Drifting modes: stochasticity



$k = \sigma a H \ll aH$

Coarse-graining exit:
Stochastic kick

Stochastic inflation

$$\dot{\bar{\phi}} = \bar{\pi} + \xi_\phi \ , \quad \dot{\bar{\pi}} = -\left(3 - \frac{1}{2}\bar{\pi}^2\right)\left(\bar{\pi} + \frac{V_{,\bar{\phi}}(\bar{\phi})}{V(\bar{\phi})}\right) + \xi_\pi$$

e-folds as time variable

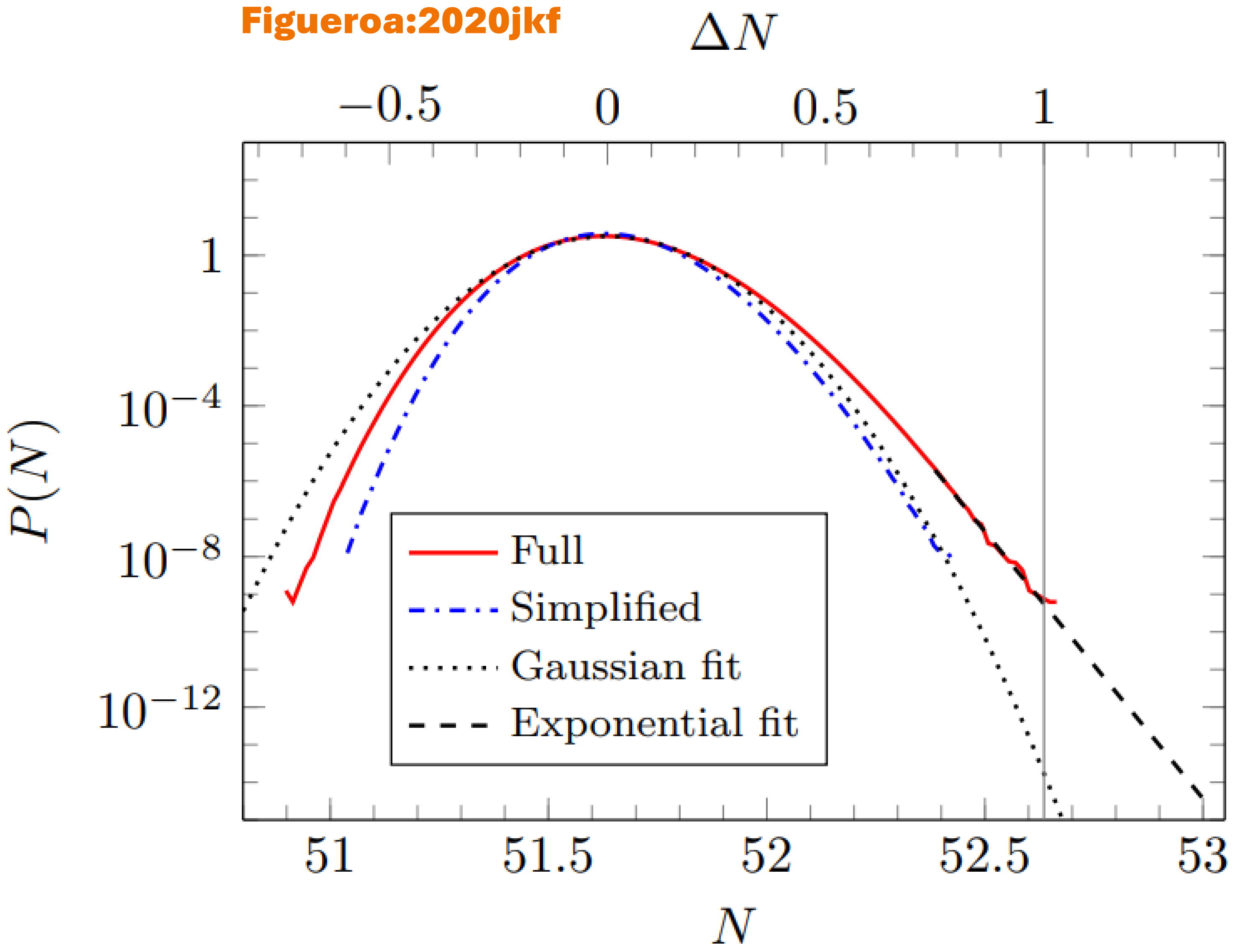
$$\langle \xi_X(N)\xi_Y(\tilde{N})\rangle = \mathcal{P}_{XY}(N, k_\sigma)\delta(N - \tilde{N})$$

↑

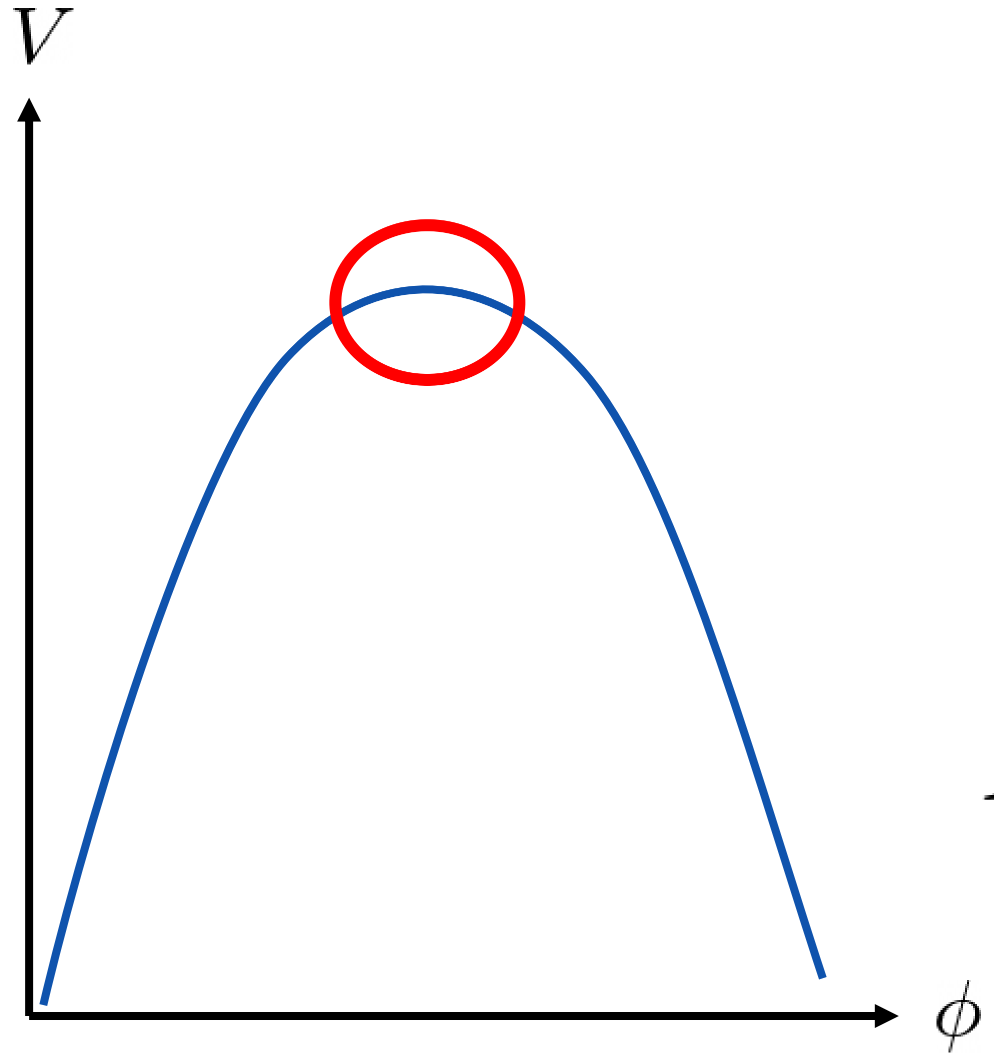
de Sitter result / numerical computation

Exponential tails

Figueroa:2020jkf



Simplification: constant roll



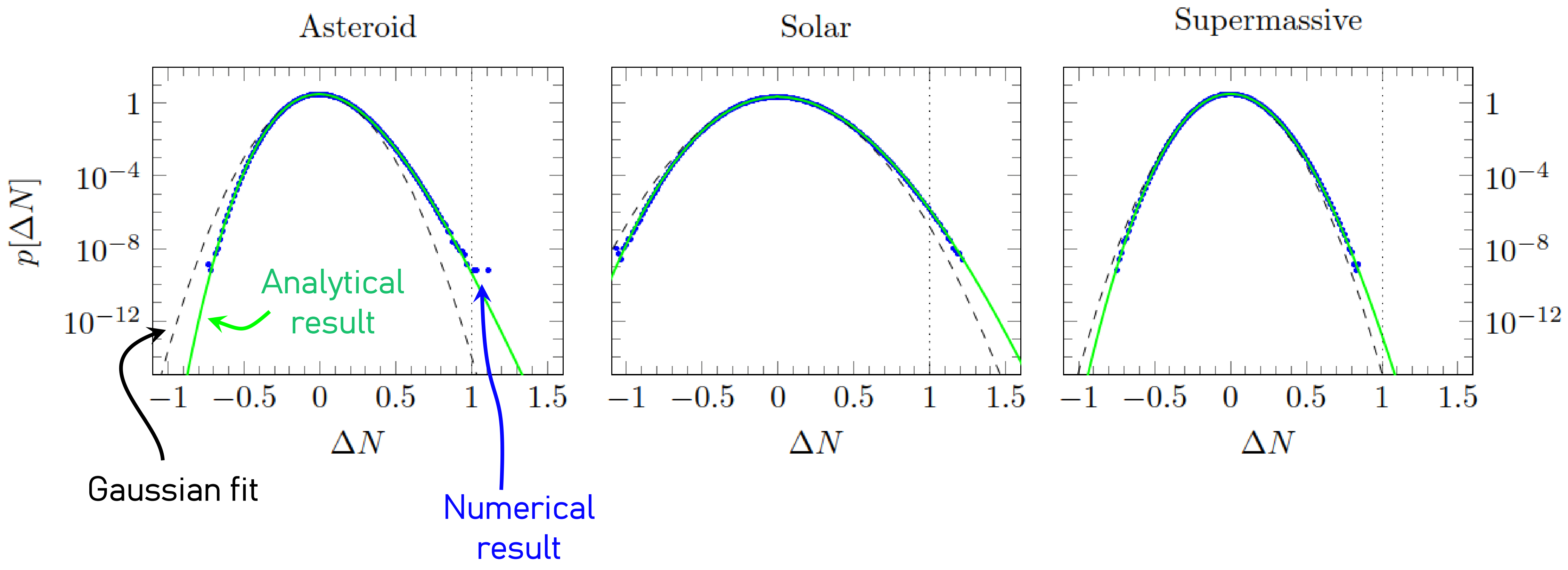
ϵ_2 approx. constant

$$\zeta = \Delta N = \frac{2}{\epsilon_2} \ln\left(1 - \frac{\epsilon_2}{2} X\right)$$

$$X = -\sum_k \sqrt{\mathcal{P}_\zeta(k) \mathrm{d} \ln k}\, \hat{\xi}_k$$

Simplification: constant roll

Tomberg:2023kli



Radial profiles

Vary coarse-graining scale for same patch: include kicks up to scale k

ΔN formalism:

$$\Delta N_{<k}(\mathbf{x}) = \zeta_{<k}(\mathbf{x}) \equiv \int \frac{\mathrm{d}^3 p}{(2\pi)^{3/2}} \zeta_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} \theta(k-p)$$

Radial profiles

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Obtain: $\zeta_k = \sqrt{\frac{\pi}{2}} \frac{1}{k^3} \frac{\mathrm{d}\zeta_{<k}}{\mathrm{d}\ln k}$ ➡ $\zeta(r) = \int_0^\infty \frac{\mathrm{d}k}{k} \frac{\mathrm{d}\zeta_{<k}}{\mathrm{d}\ln k} \frac{\sin(kr)}{kr}$

RECAP

Gaussian ζ from linear perturbations

(+ mean profiles)

↓

Stochastic inflation: non-Gaussian tails

↓

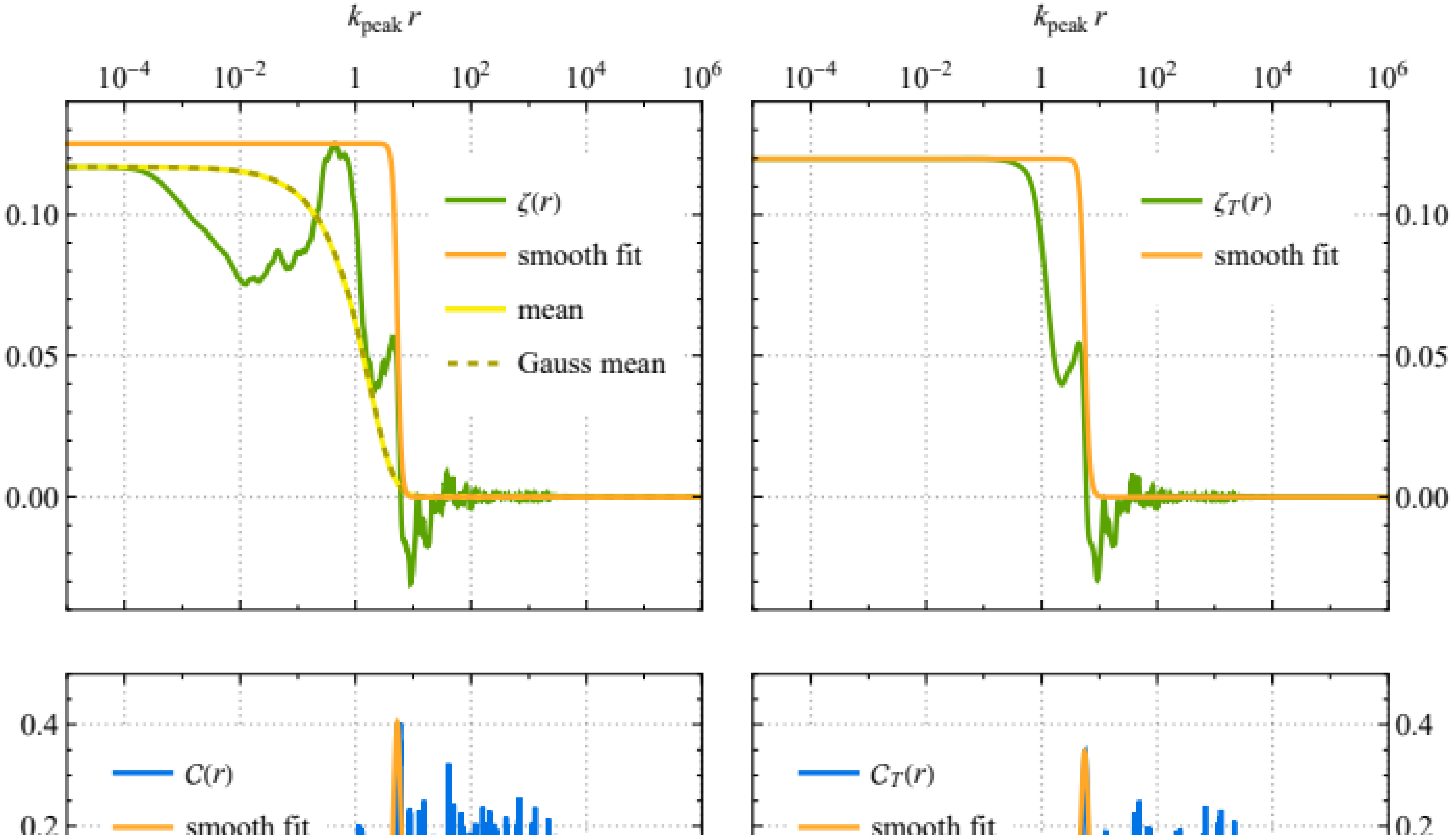
Varying coarse-graining scale

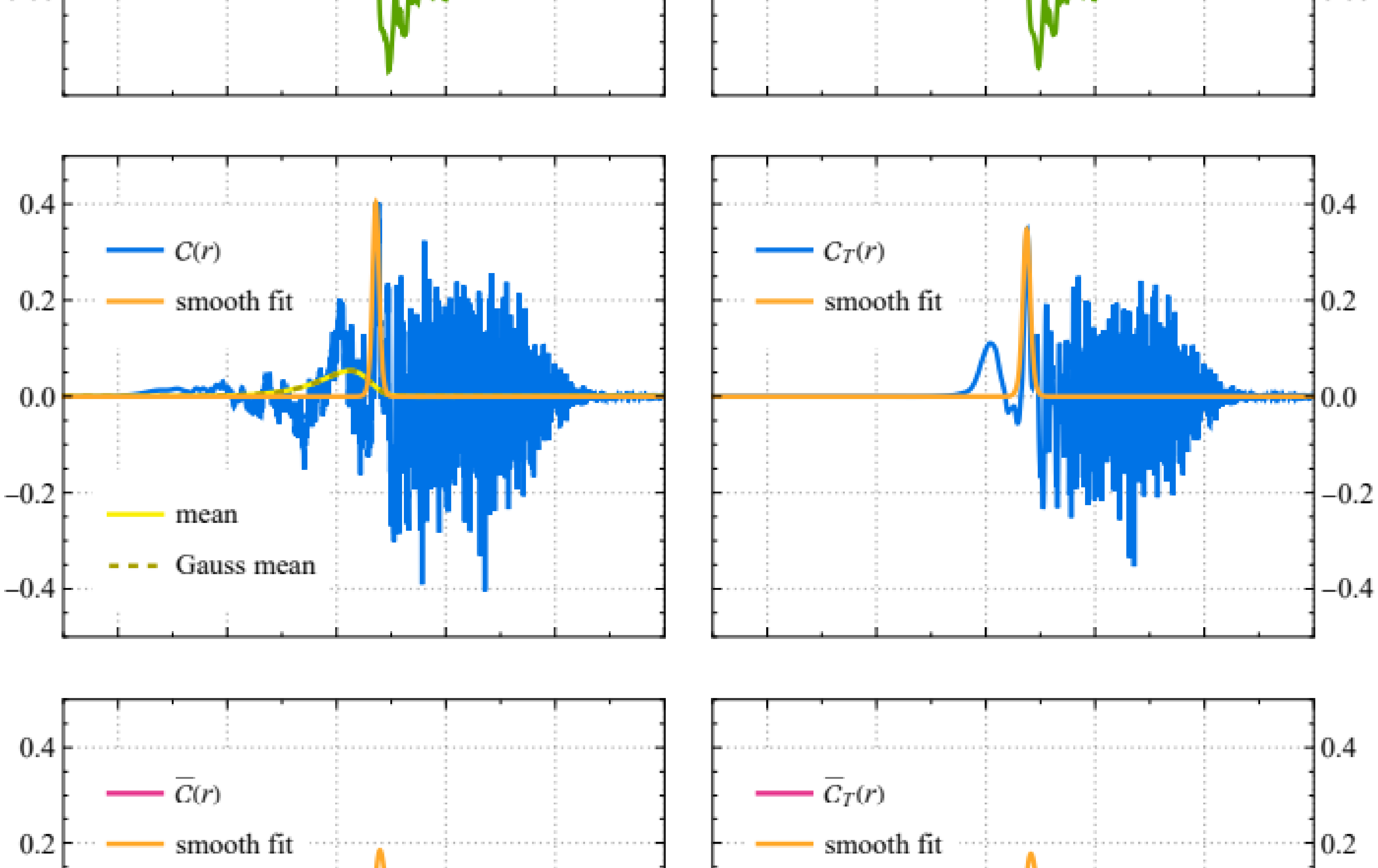
↓

Full $\mathcal{C}(r)$ profiles

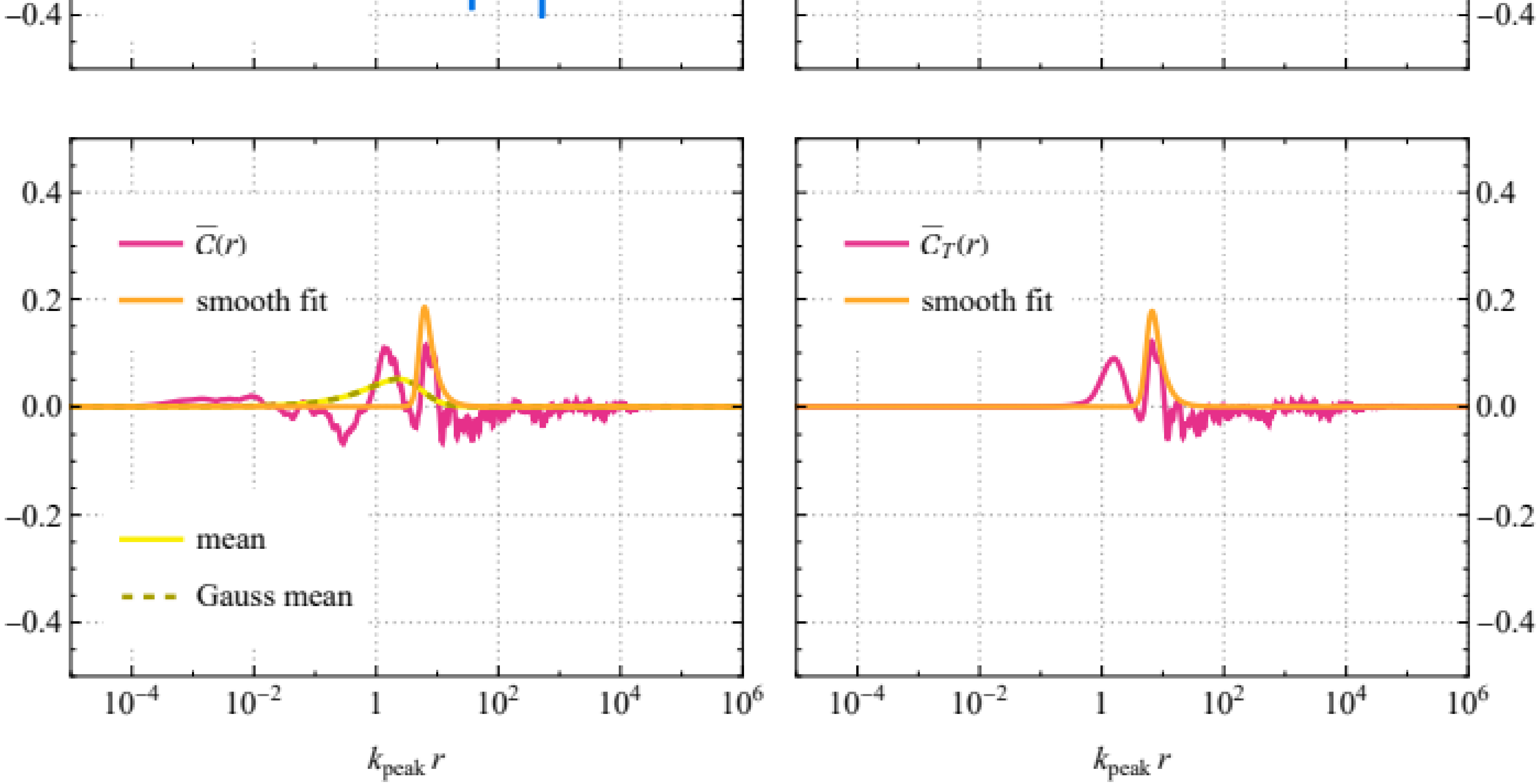
(+ constant-roll approx.)

Raatikainen:2025gpd

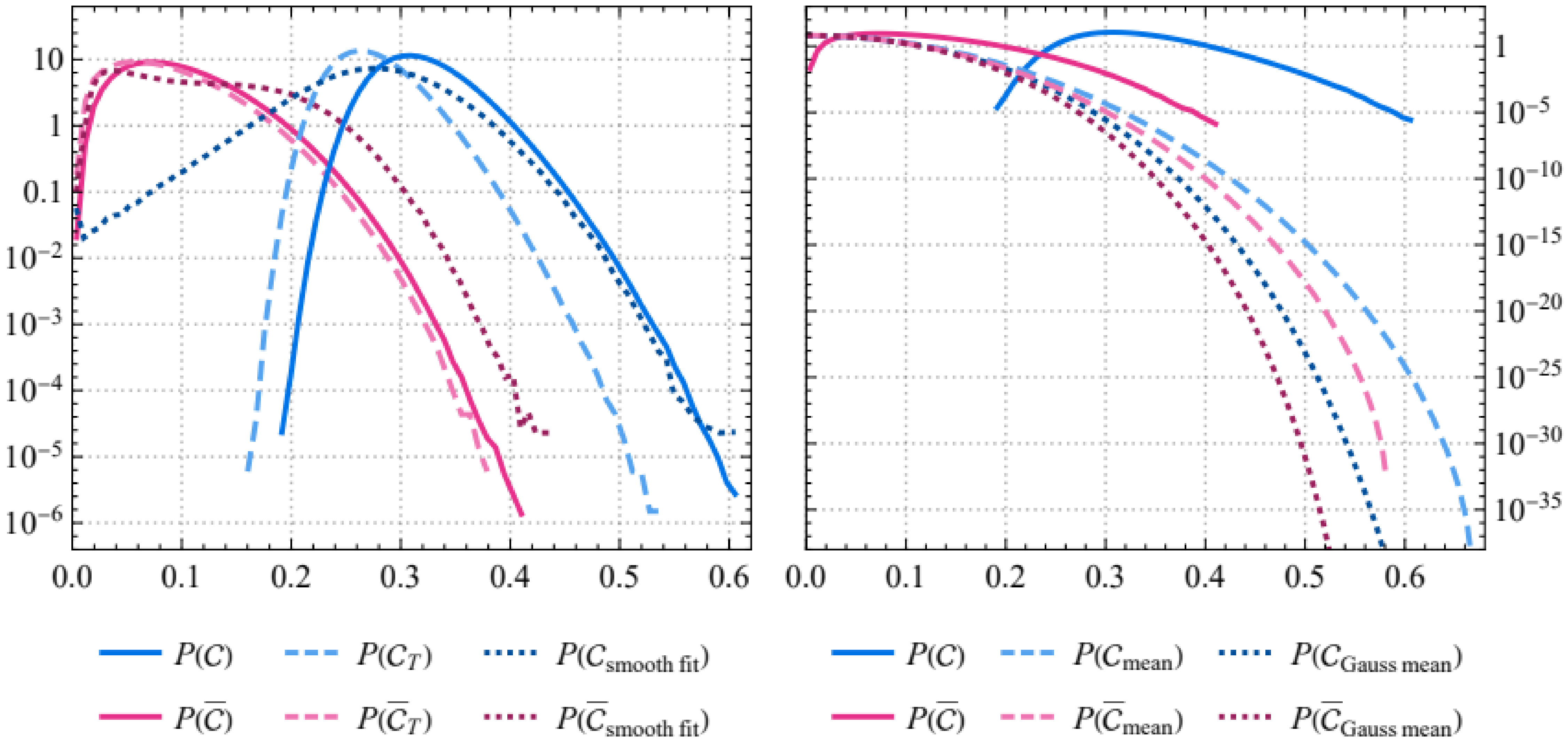




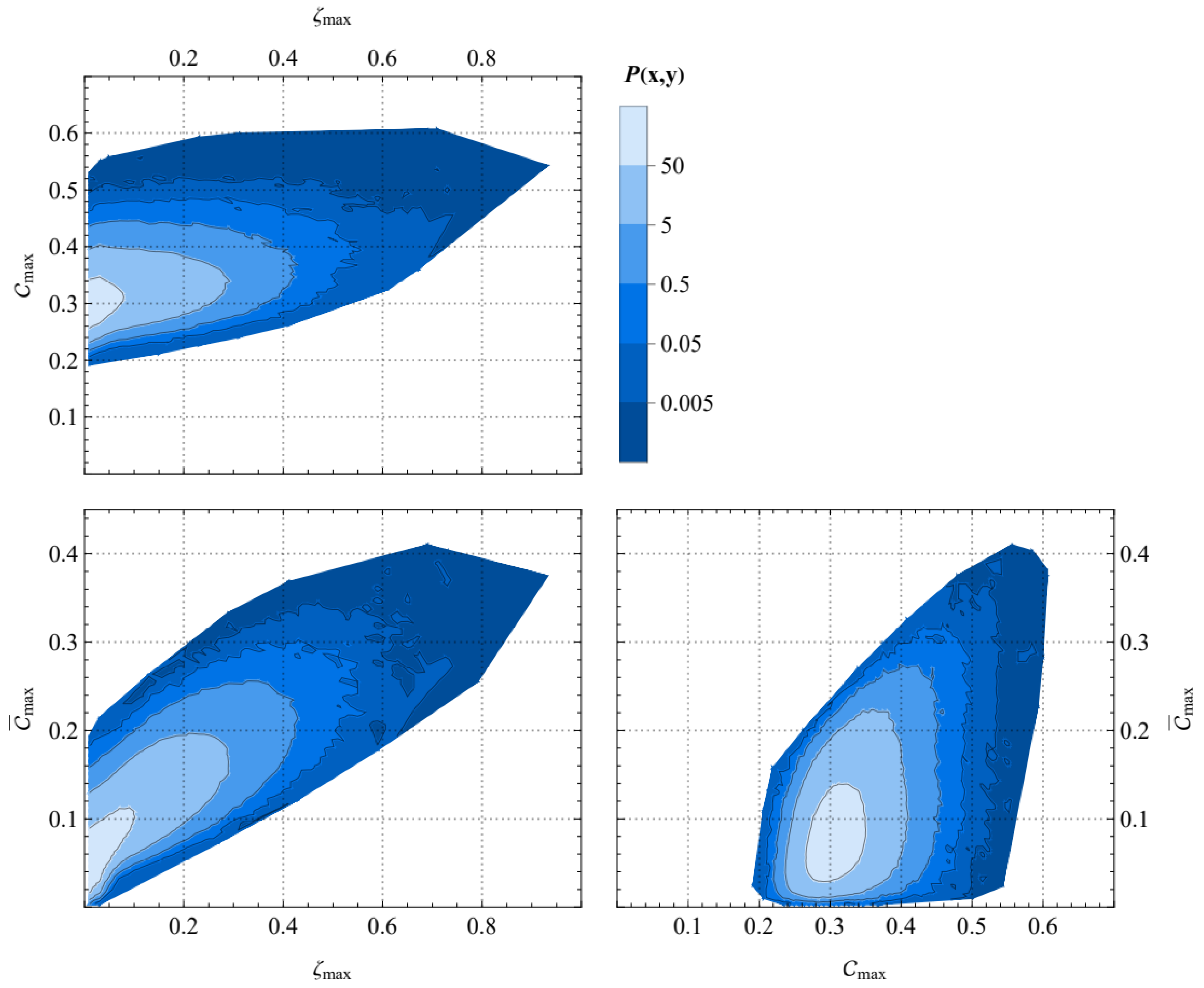
0.4
0.2
0.0
−0.2
−0.4
$C(r)$
smooth fit
mean
Gauss mean
$C_T(r)$
smooth fit
$\overline{C}(r)$
smooth fit
$\overline{C}_T(r)$
smooth fit



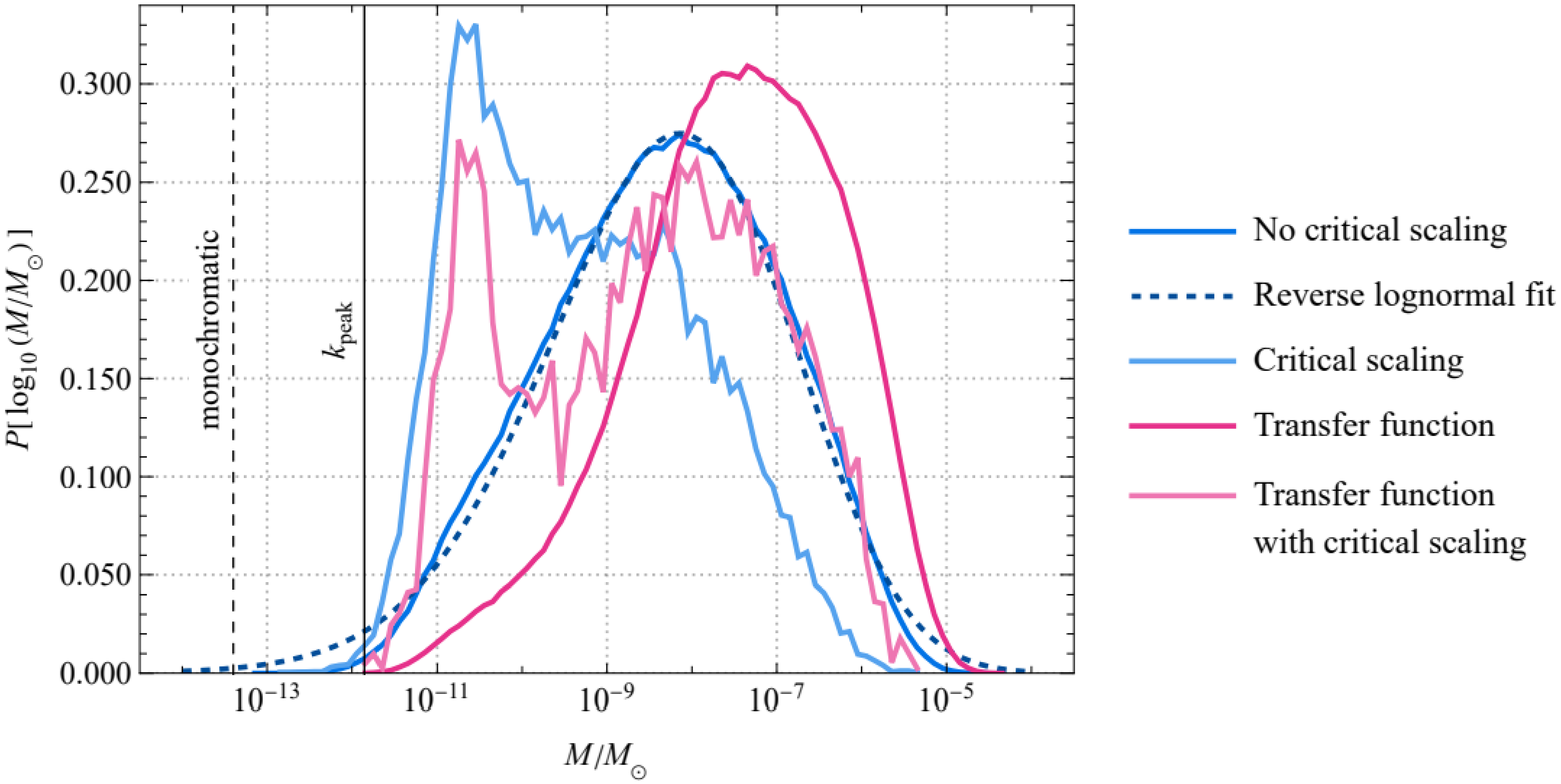
$\overline{C}(r)$
smooth fit
mean
Gauss mean
$\overline{C}_T(r)$
smooth fit
$k_{\rm peak}\, r$
$k_{\rm peak}\, r$



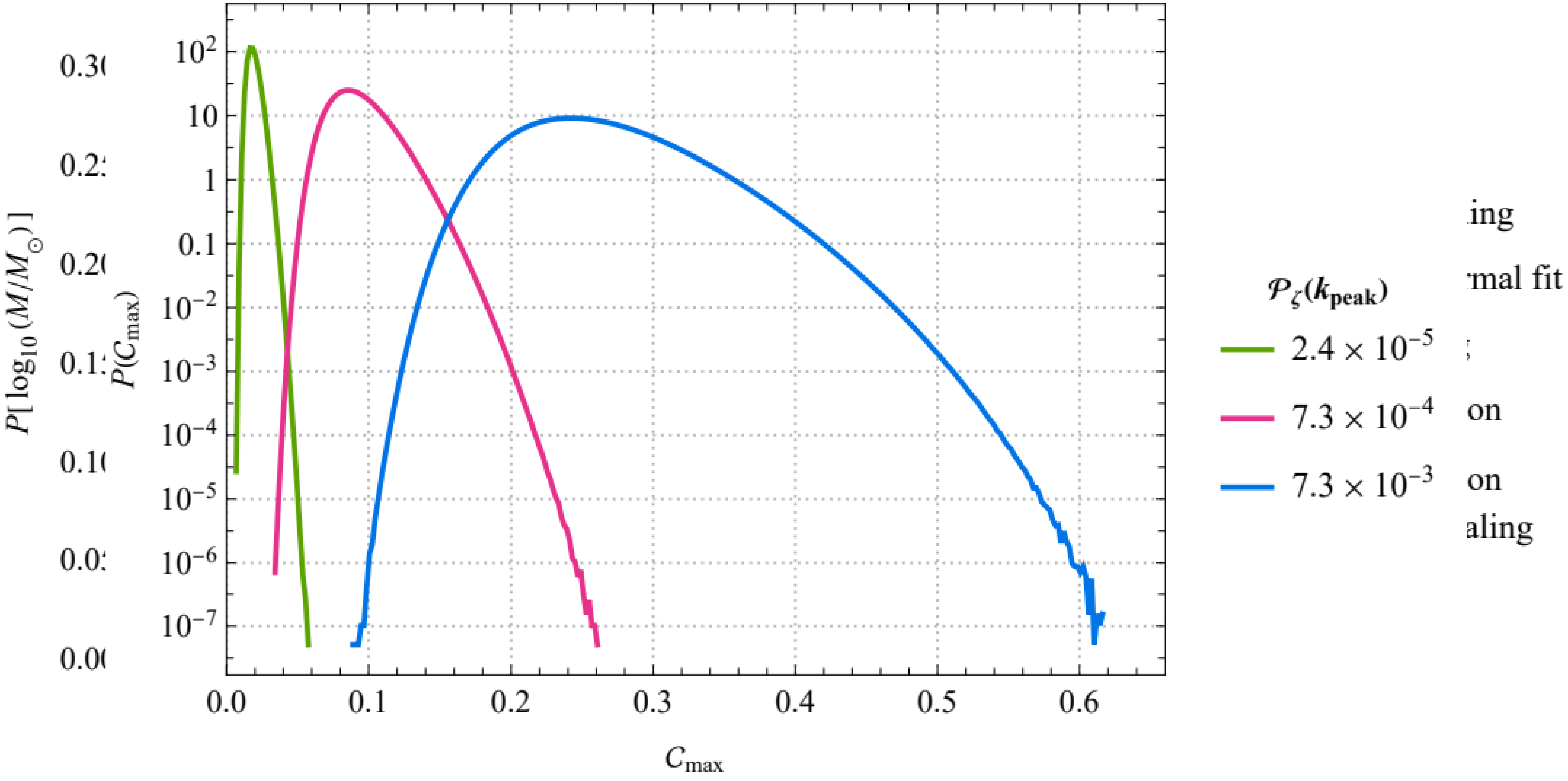
10
1
0.1
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
0.0
0.1
0.2
0.3
0.4
0.5
0.6
1
10^{-5}
10^{-10}
10^{-15}
10^{-20}
10^{-25}
10^{-30}
10^{-35}
$P(C)$
$P(C_T)$
$P(C_{\text{smooth fit}})$
$P(\overline{C})$
$P(\overline{C}_T)$
$P(\overline{C}_{\text{smooth fit}})$
$P(C)$
$P(C_{\text{mean}})$
$P(C_{\text{Gauss mean}})$
$P(\overline{C})$
$P(\overline{C}_{\text{mean}})$
$P(\overline{C}_{\text{Gauss mean}})$



ζ_{max}
0.2
0.4
0.6
0.8
P(x,y)
50
5
0.5
0.05
0.005
C_{max}
0.6
0.5
0.4
0.3
0.2
0.1
$\overline{C}_{max}$
0.4
0.3
0.2
0.1
ζ_{max}
C_{max}



0.300
0.250
0.200
0.150
0.100
0.050
0.000
$P[\log_{10}(M/M_\odot)]$
monochromatic
$k_{\rm peak}$
10^{-13}
10^{-11}
10^{-9}
10^{-7}
10^{-5}
$M/M_\odot$
No critical scaling
Reverse lognormal fit
Critical scaling
Transfer function
Transfer function with critical scaling



$P(C_{\max})$
10^2
10
1
0.1
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
0.0
0.1
0.2
0.3
0.4
0.5
0.6
$C_{\max}$
$\mathcal{P}_\zeta(k_{\text{peak}})$
2.4×10^{-5}
7.3×10^{-4}
7.3×10^{-3}
$P[\log_{10}(M/M_\odot)]$

Open questions

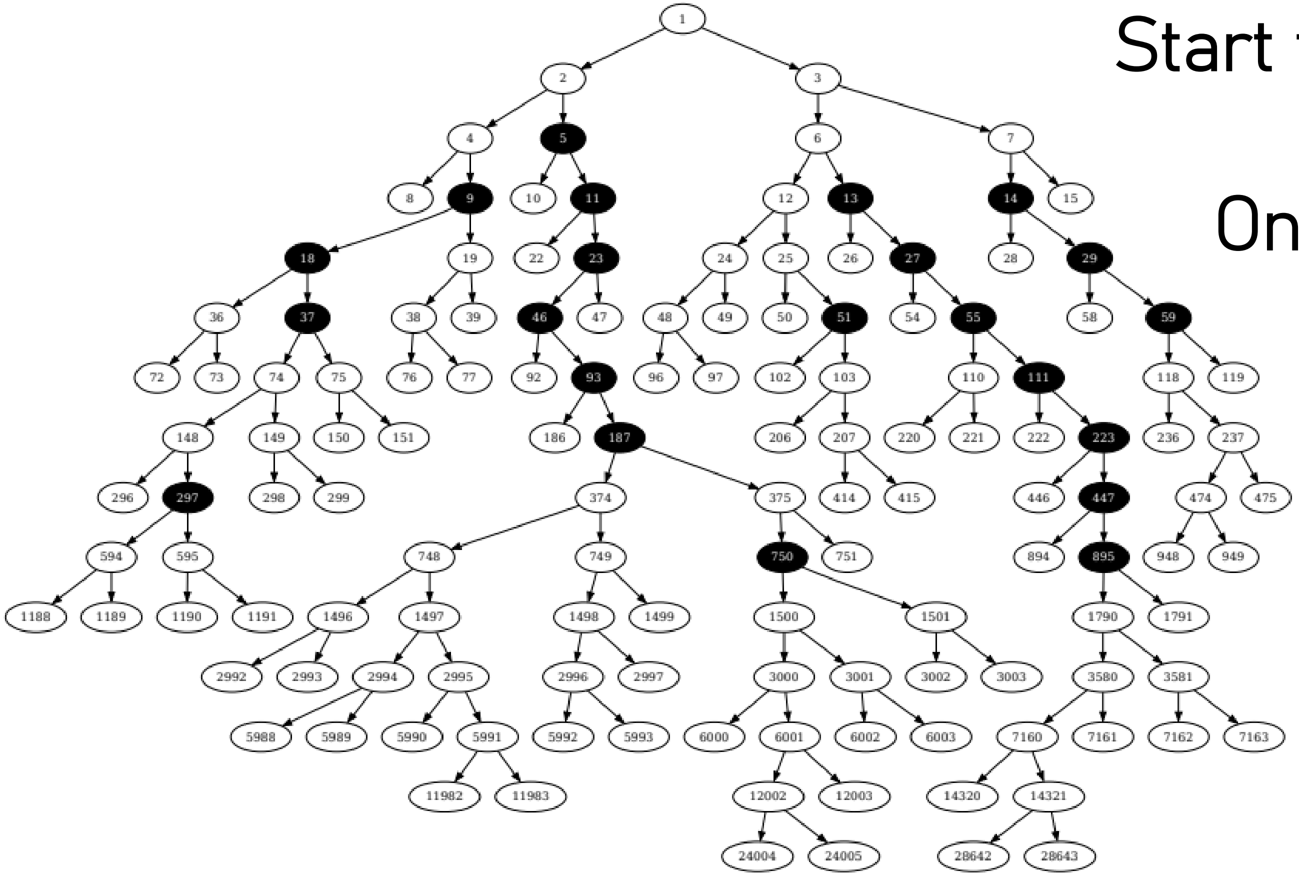
Convergence at large radius?

Correct collapse criterion for spiky profiles?

Redo collapse simulations?

Stochastic trees

Animali:2025pyf



Start from 1 Hubble patch

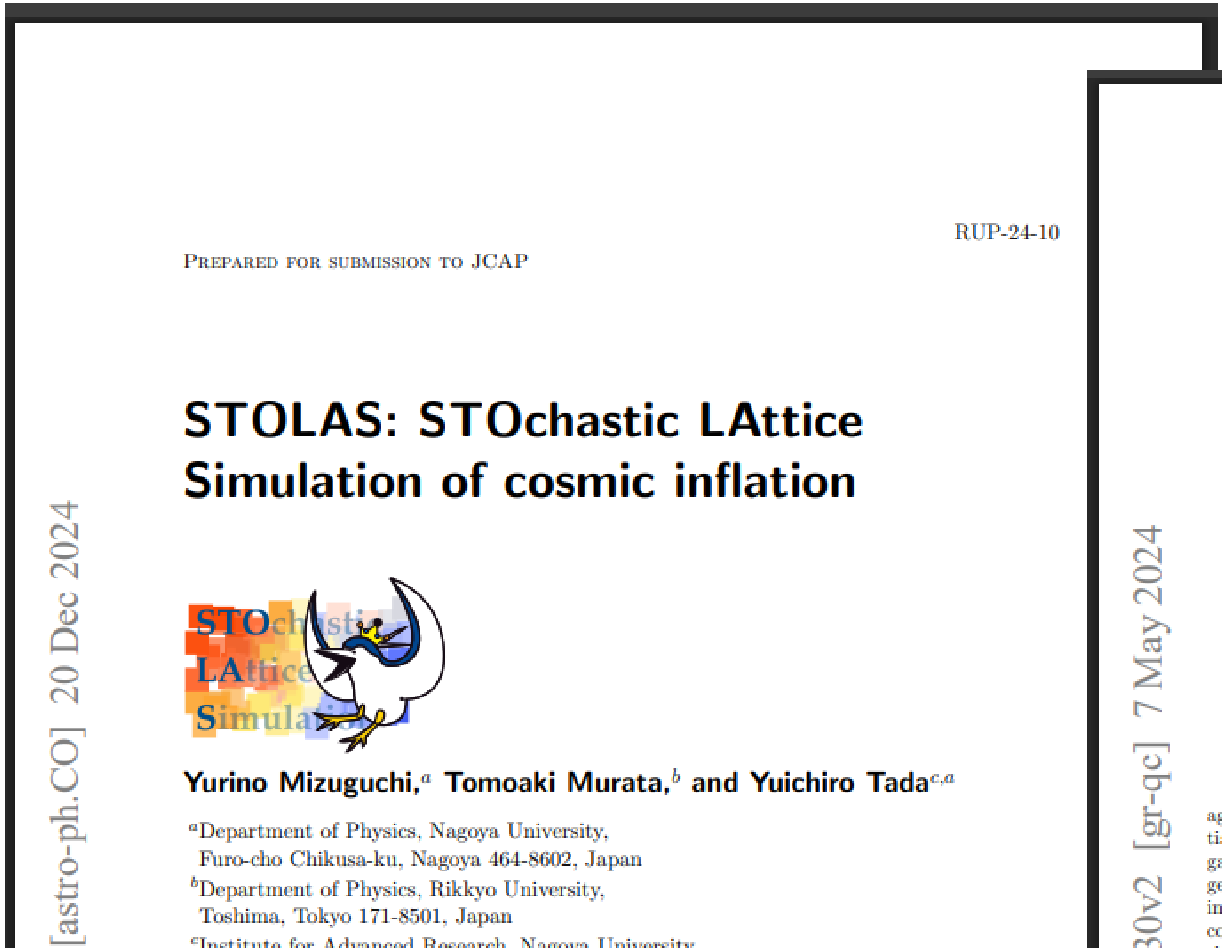
Once volume doubled: split in two

Spatial distribution?
Clustering?

Lattice simulations

Mizuguchi:2024kbl

Stochastic grid



RUP-24-10

Prepared for submission to JCAP

STOLAS: STOchastic LAttice Simulation of cosmic inflation

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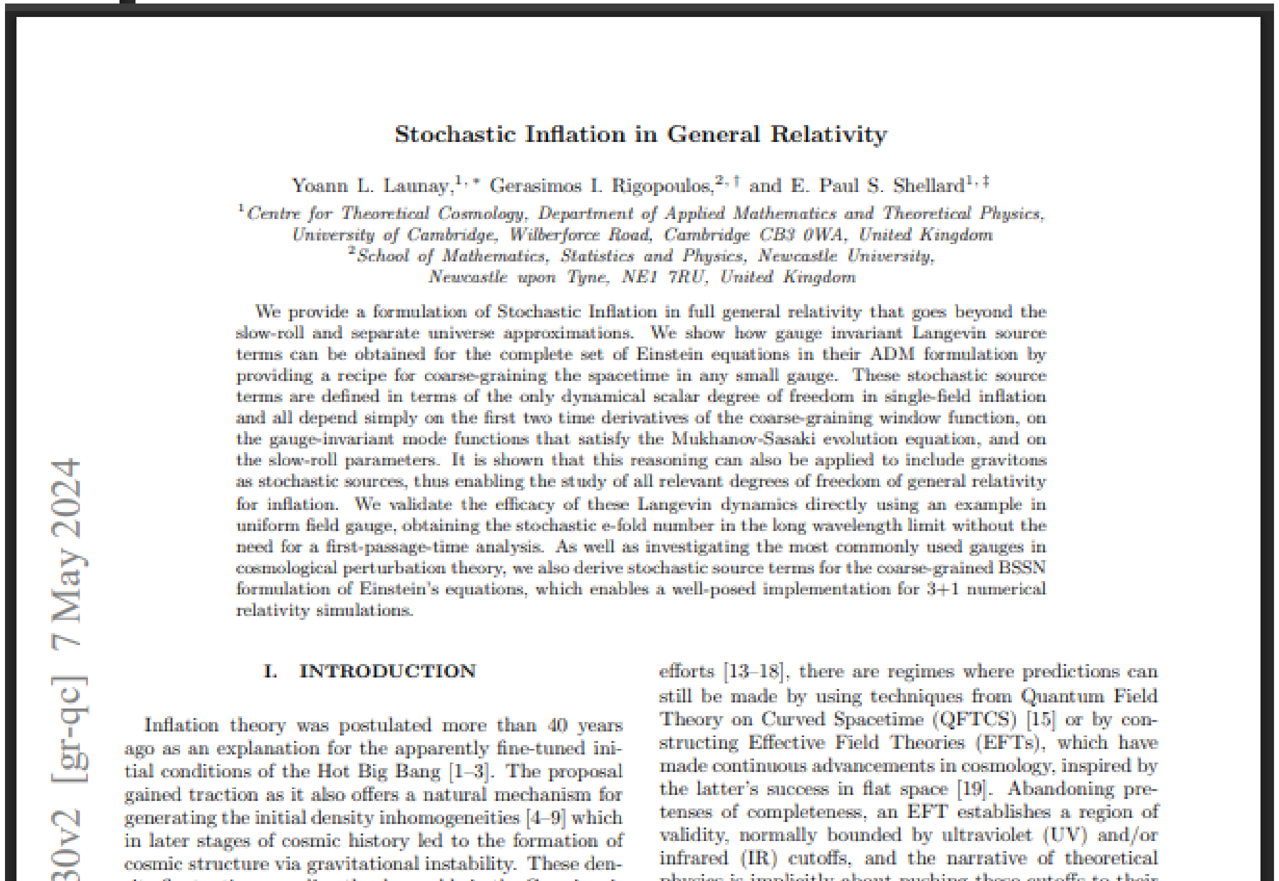
[c]Institute for Advanced Research, Nagoya University

Launay:2024qsm

Combining stochastics and numerical relativity



Stochastic Inflation in General Relativity

Yoann L. Launay,[1,*] Gerasimos I. Rigopoulos,[2,†] and E. Paul S. Shellard[1,‡]

[1]*Centre for Theoretical Cosmology, Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom*
[2]*School of Mathematics, Statistics and Physics, Newcastle University, Newcastle upon Tyne, NE1 7RU, United Kingdom*

We provide a formulation of Stochastic Inflation in full general relativity that goes beyond the slow-roll and separate universe approximations. We show how gauge invariant Langevin source terms can be obtained for the complete set of Einstein equations in their ADM formulation by providing a recipe for coarse-graining the spacetime in any small gauge. These stochastic source terms are defined in terms of the only dynamical scalar degree of freedom in single-field inflation and all depend simply on the first two time derivatives of the coarse-graining window function, on the gauge-invariant mode functions that satisfy the Mukhanov-Sasaki evolution equation, and on the slow-roll parameters. It is shown that this reasoning can also be applied to include gravitons as stochastic sources, thus enabling the study of all relevant degrees of freedom of general relativity for inflation. We validate the efficacy of these Langevin dynamics directly using an example in uniform field gauge, obtaining the stochastic e-fold number in the long wavelength limit without the need for a first-passage-time analysis. As well as investigating the most commonly used gauges in cosmological perturbation theory, we also derive stochastic source terms for the coarse-grained BSSN formulation of Einstein's equations, which enables a well-posed implementation for 3+1 numerical relativity simulations.

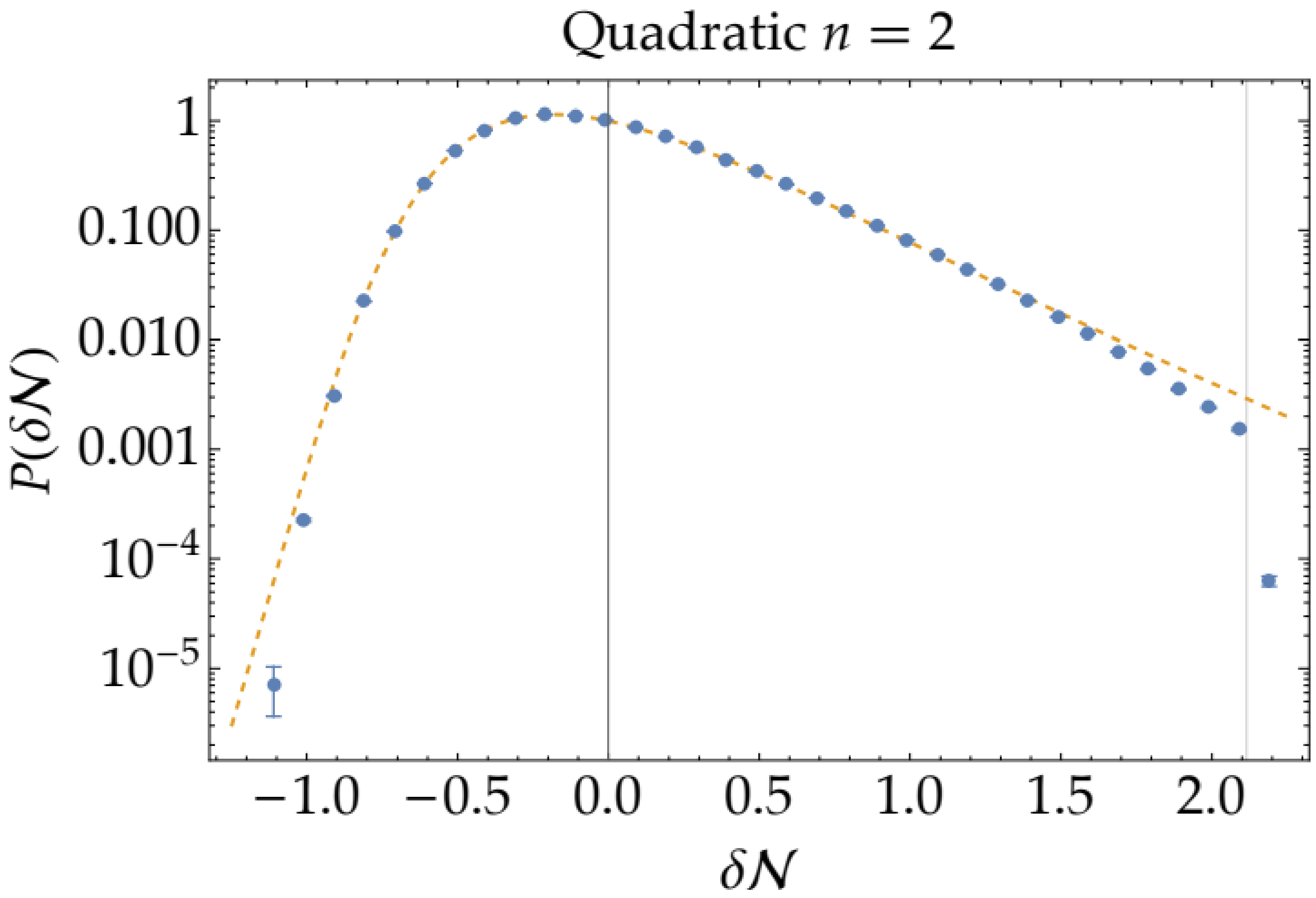
I. INTRODUCTION

Inflation theory was postulated more than 40 years ago as an explanation for the apparently fine-tuned initial conditions of the Hot Big Bang [1–3]. The proposal gained traction as it also offers a natural mechanism for generating the initial density inhomogeneities [4–9] which in later stages of cosmic history led to the formation of cosmic structure via gravitational instability. These den-

efforts [13–18], there are regimes where predictions can still be made by using techniques from Quantum Field Theory on Curved Spacetime (QFTCS) [15] or by constructing Effective Field Theories (EFTs), which have made continuous advancements in cosmology, inspired by the latter's success in flat space [19]. Abandoning pretenses of completeness, an EFT establishes a region of validity, normally bounded by ultraviolet (UV) and/or infrared (IR) cutoffs, and the narrative of theoretical

Multi-field inflation

Murata:2025onc



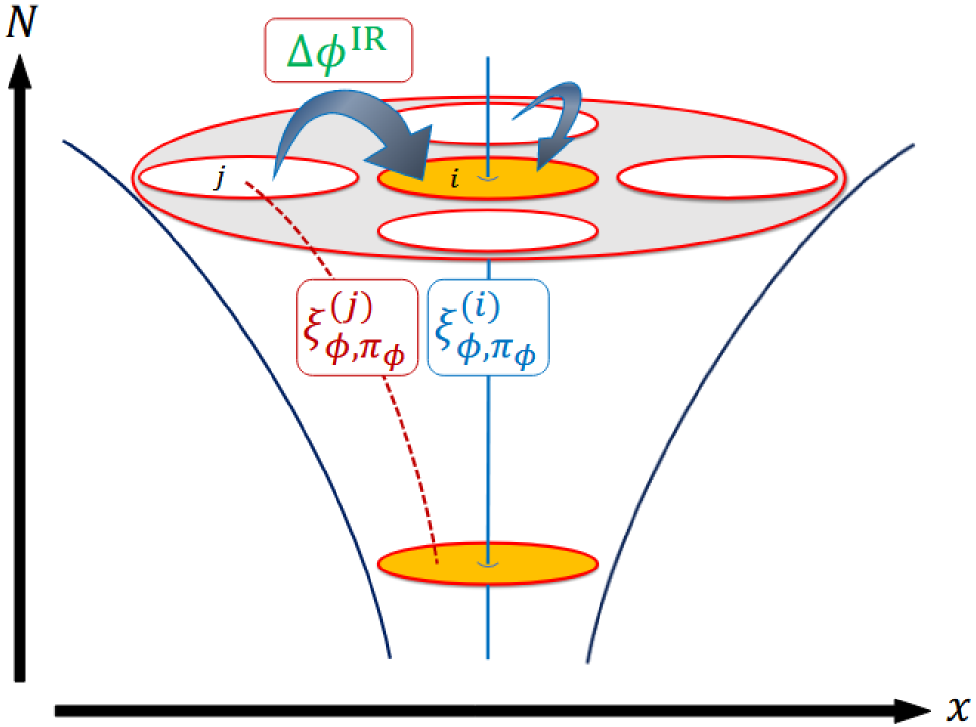
Hybrid inflation model

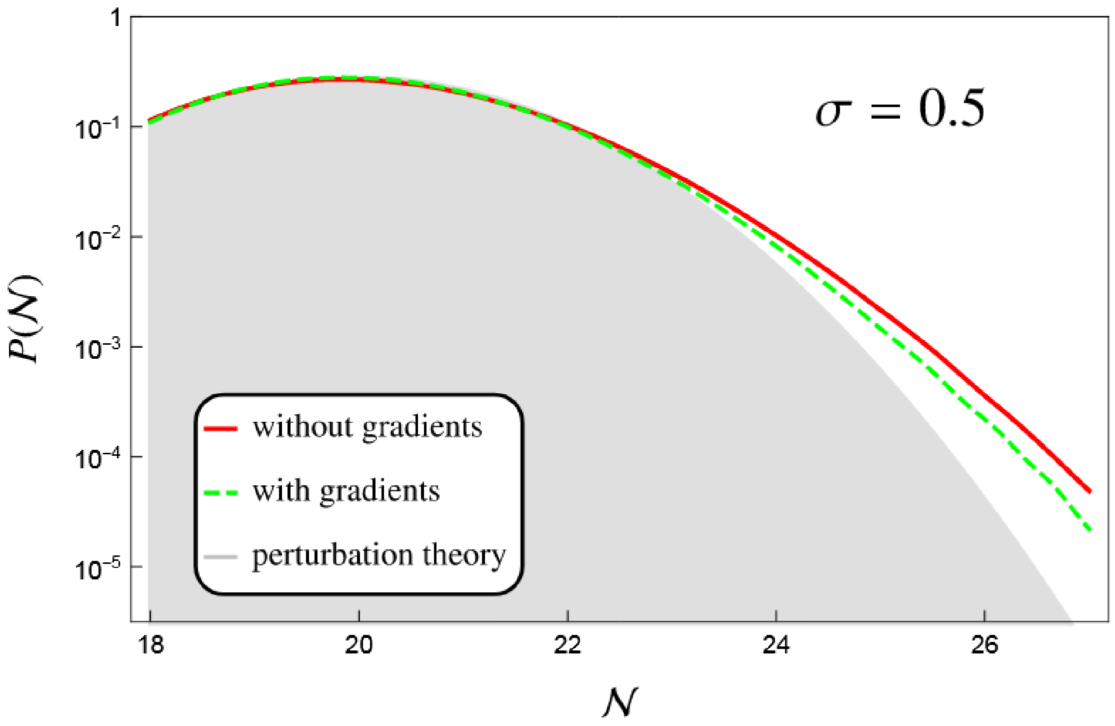
Upper bound to curvature perturbation

Gradient corrections

Briaud:2025ayt

Go beyond leading order in gradient expansion





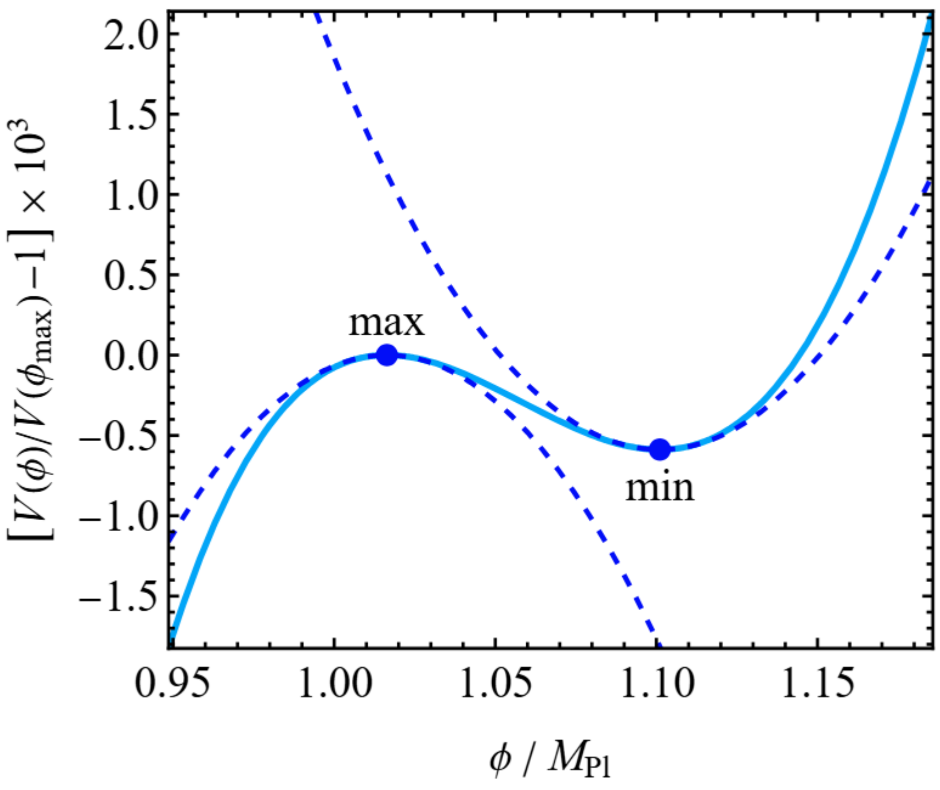
Colored noise

Exponential tails: a pullback effect

Eternal inflation

Tomberg:2025fku

$$\text{eternal inflation} \quad \Longleftrightarrow \quad \lim_{N\to\infty} \langle V \rangle_N \equiv \lim_{N\to\infty} \int_{\phi_{b1}}^{\phi_{b2}} e^{3N} P(\phi, N) \mathrm{d}\phi > 0$$



$$P(\phi, N) \sim e^{-\lambda N} \qquad \lambda \leq 3$$

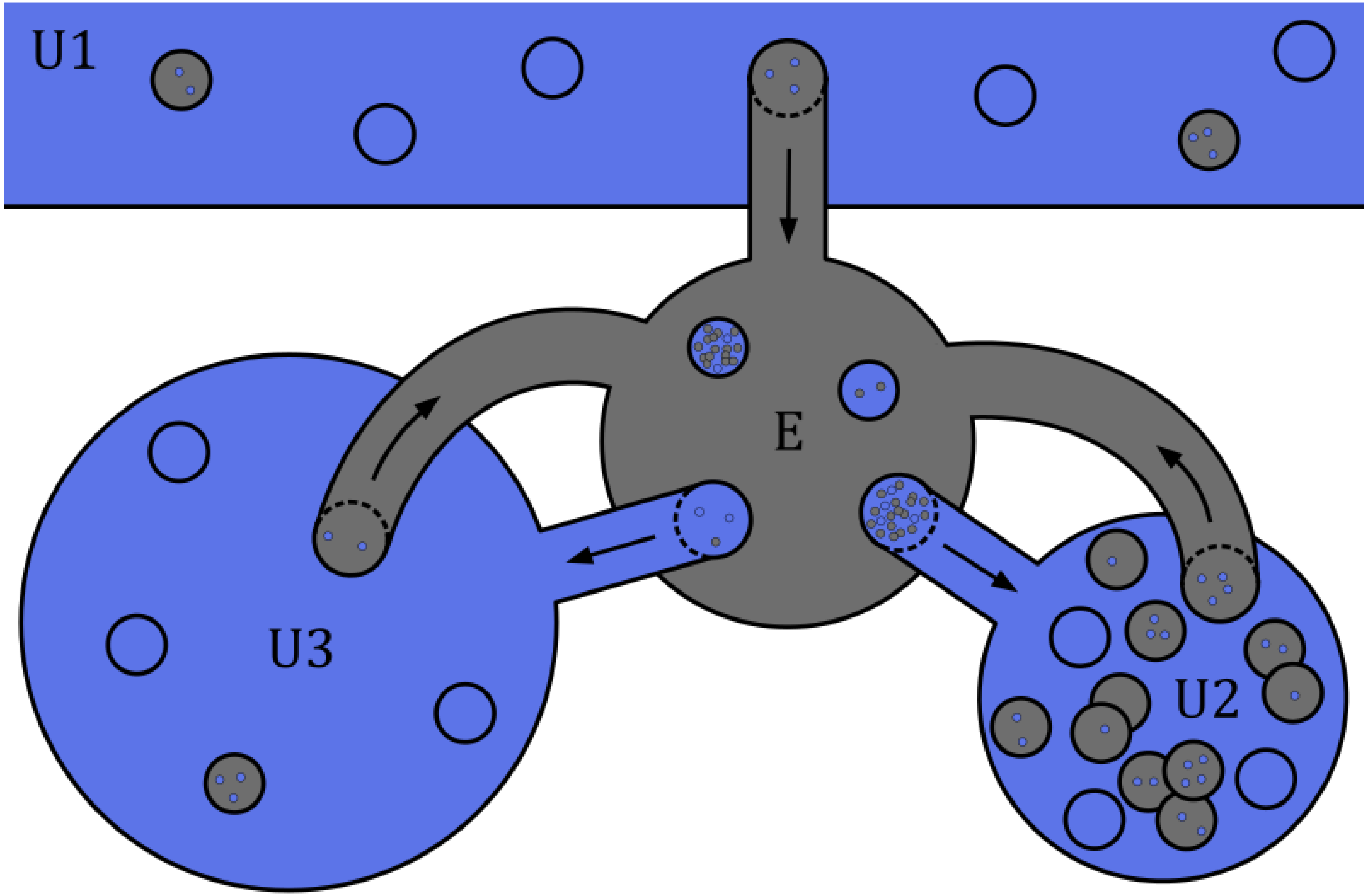
Maximum:

$$\lambda \approx |\eta_H| \sim 0.1$$

Minimum:

$$\lambda \approx 0$$

Eternal inflation



SUMMARY

Stochastic inflation provides
radial curvature profiles

Curvature profiles
give compaction function

Profiles very spiky:
relation to collapse simulations?