Primordial black holes
and
stochastic inflation

University of Nottingham, 15 October 2024

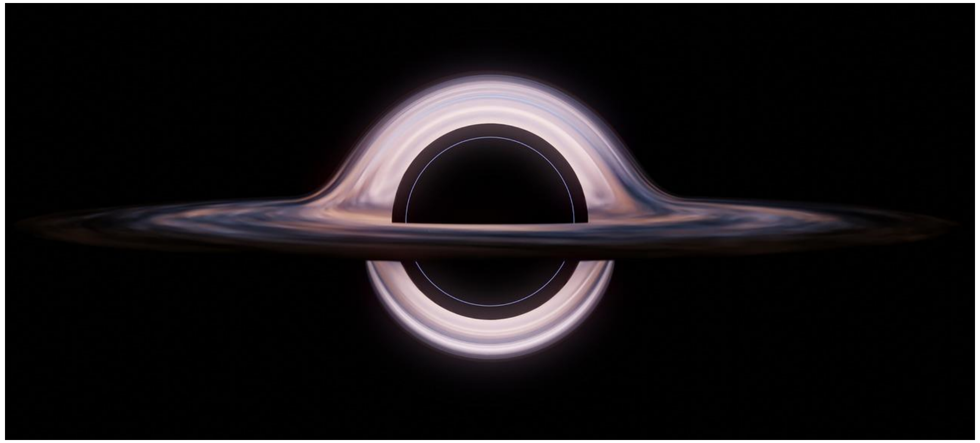
Eemeli Tomberg, Lancaster University

Based on
2012.06551, 2111.07437, 2210.17441, 2304.10903, 2312.12911, 2409.12950
in collaboration with D. Figueroa, S. Raatikainen, S. Räsänen, et al

Why primordial black holes (PBHs)?

Black holes formed in early Universe

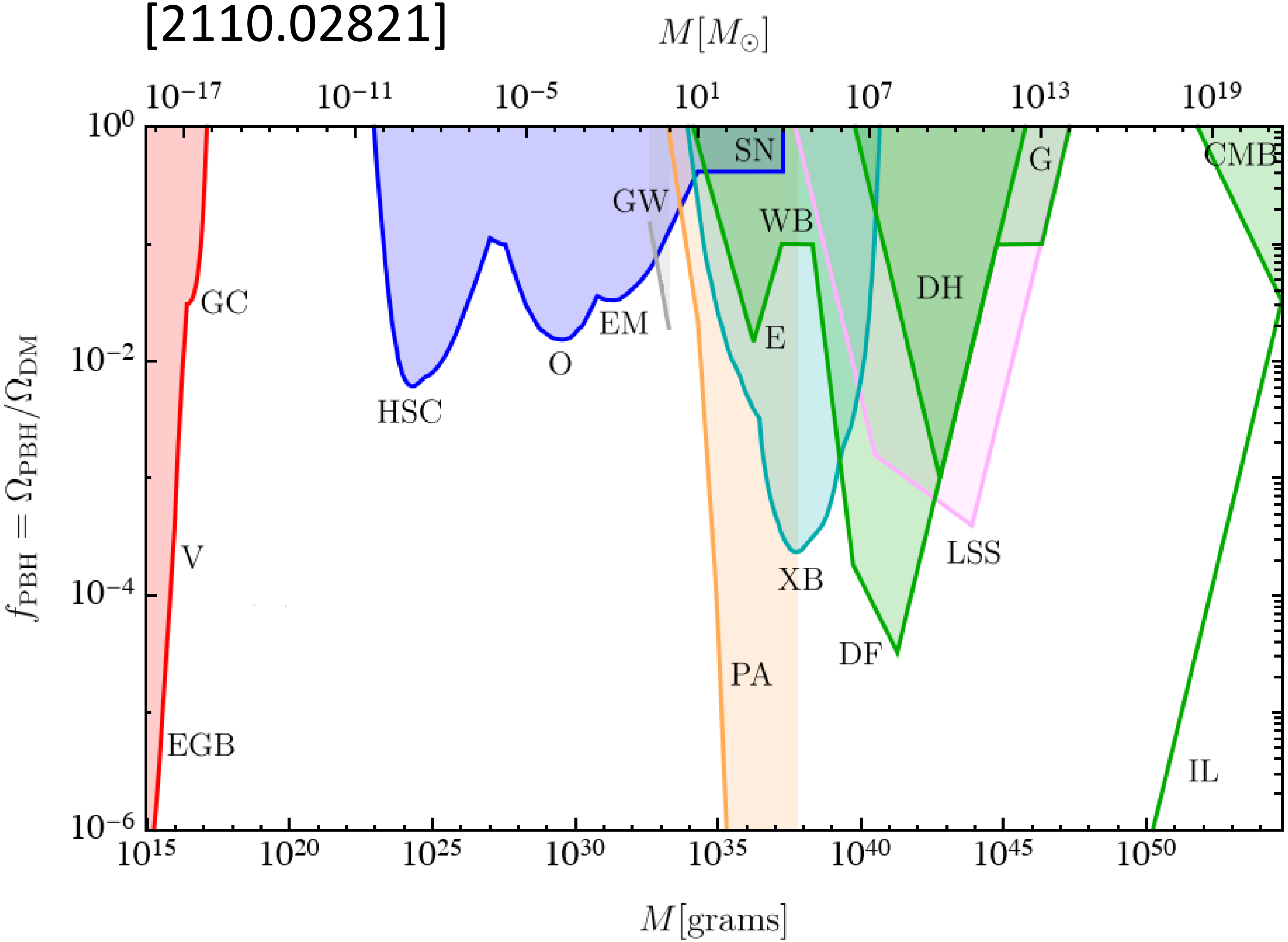
- Carry information of conditions there (small-scale perturbations)
- Any mass (Hawking evaporation?)



Applications in cosmology

- Dark matter candidate
- Seeds of supermassive black holes
- GW source

[2110.02821]

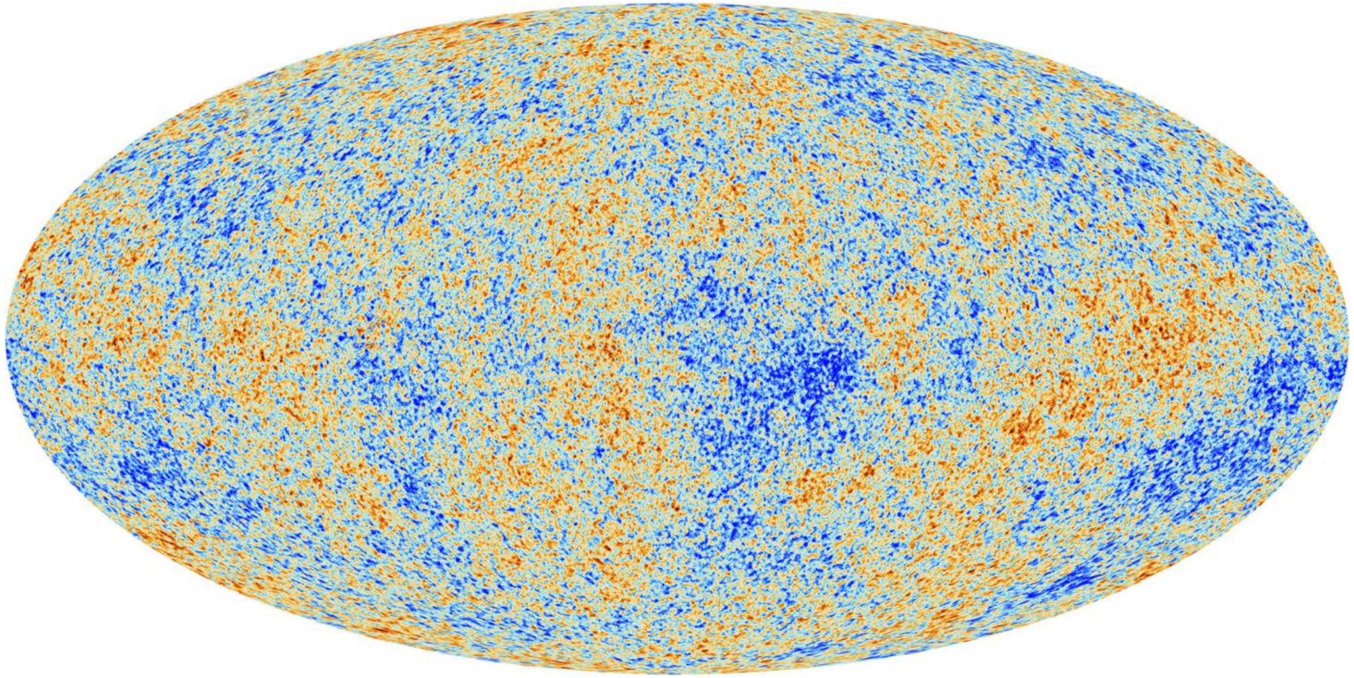


$M[M_\odot]$
10^{-17}
10^{-11}
10^{-5}
10^{1}
10^{7}
10^{13}
10^{19}
10^{0}
10^{-2}
10^{-4}
10^{-6}
$f_{\rm PBH} = \Omega_{\rm PBH}/\Omega_{\rm DM}$
GC
V
EGB
HSC
O
EM
GW
SN
WB
E
XB
PA
DF
DH
G
LSS
CMB
IL
10^{15}
10^{20}
10^{25}
10^{30}
10^{35}
10^{40}
10^{45}
10^{50}
M[grams]

Black holes from primordial perturbations

Cosmic inflation: quantum fluctuations

Later: strongest collapse into black holes



I. (Semi-)inflection point inflation

II. Stochastic inflation

III. Black hole statistics

IV. An axion-curvaton model

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Single-field inflation is simple

Action:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - V(\varphi)\right]$$

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Background equations of motion:

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0\,, \quad 3H^2 = \frac{1}{2}\dot{\varphi}^2 + V(\varphi)$$

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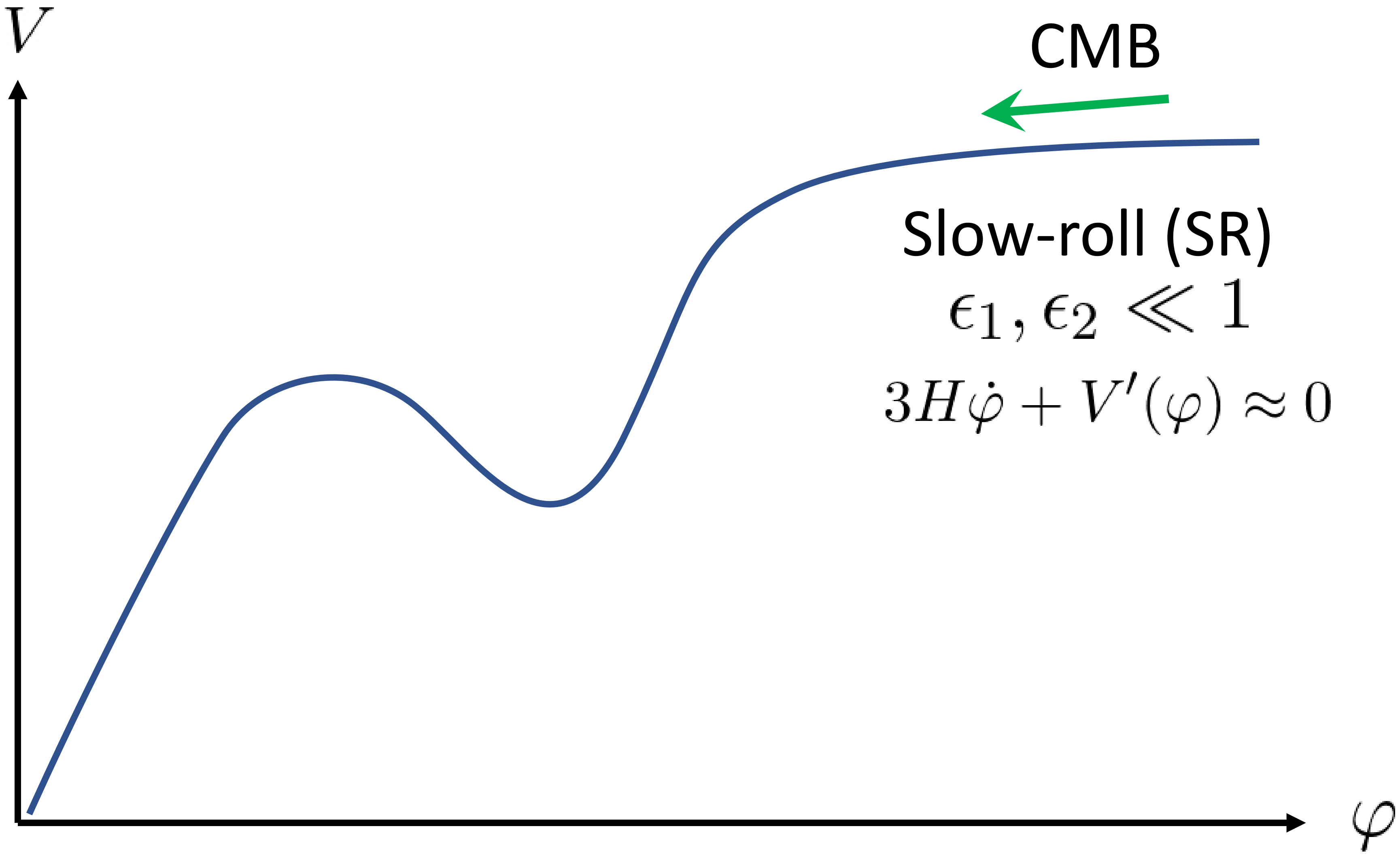
Slow-roll parameters:

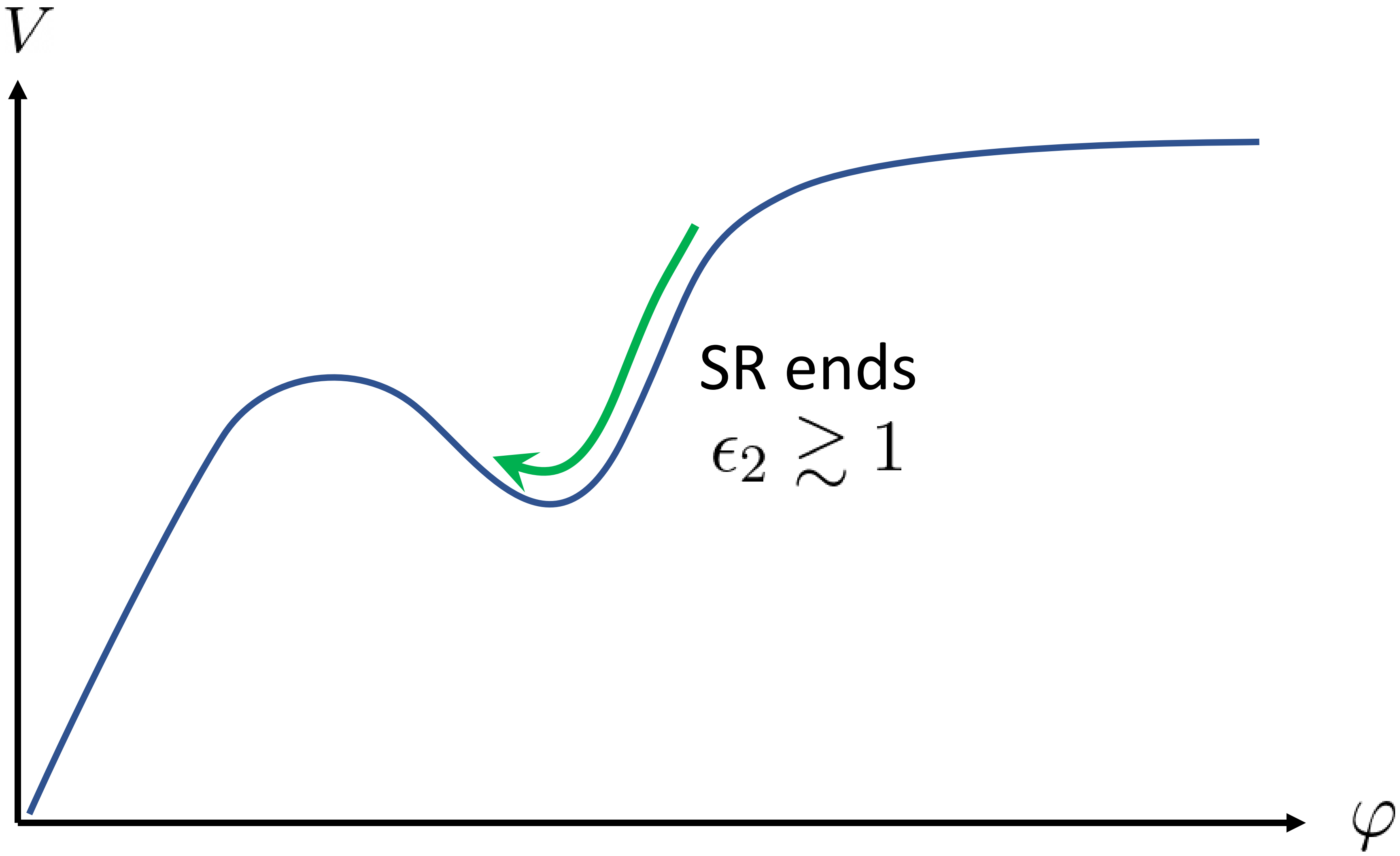
$$\epsilon_1 \equiv -\partial_N \ln H\,, \quad \epsilon_2 \equiv \partial_N \ln \epsilon_1$$



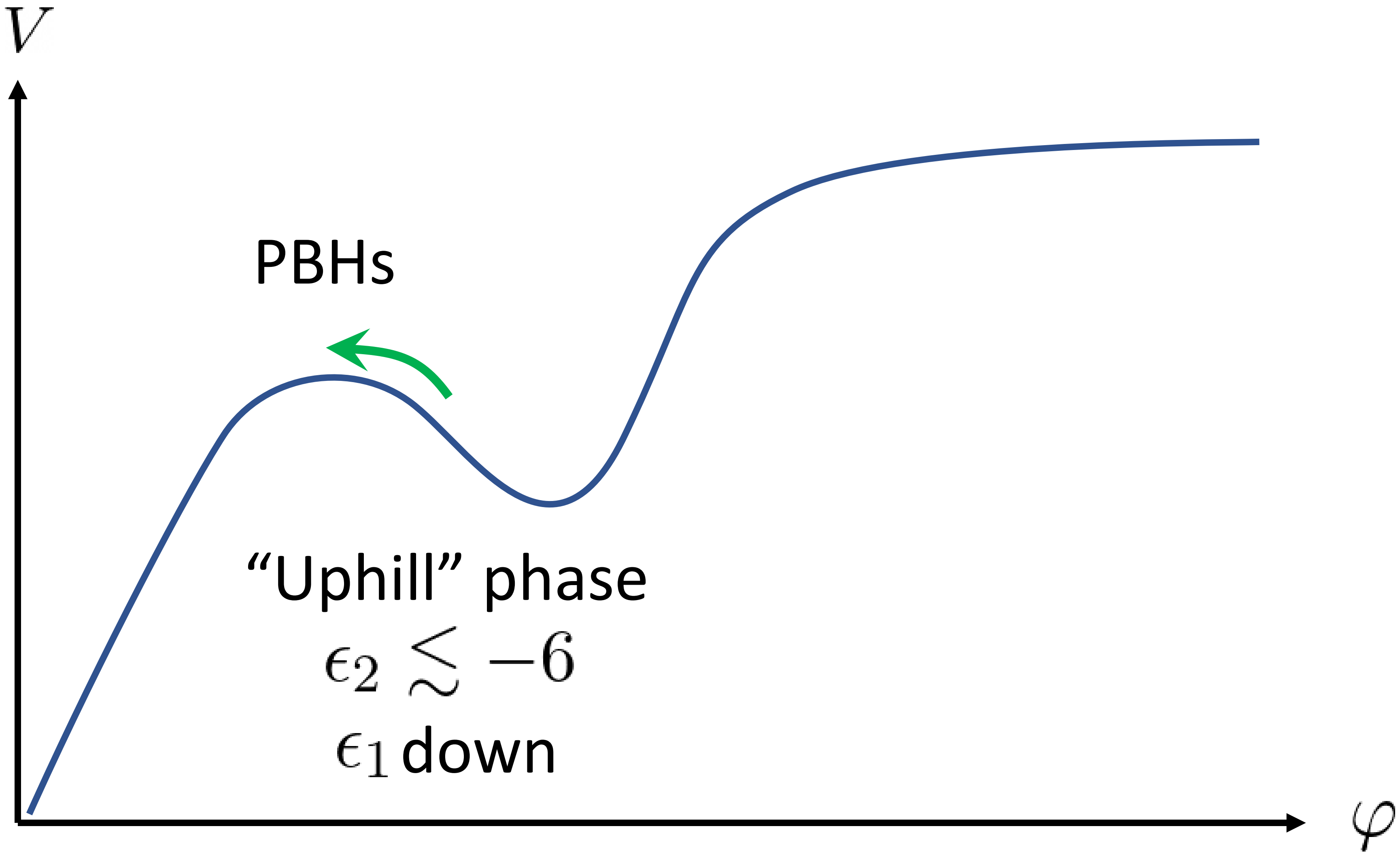
Slow-roll (SR)

$$\epsilon_1, \epsilon_2 \ll 1$$

$$3H\dot{\varphi} + V'(\varphi) \approx 0$$



SR ends
$\epsilon_2 \gtrsim 1$

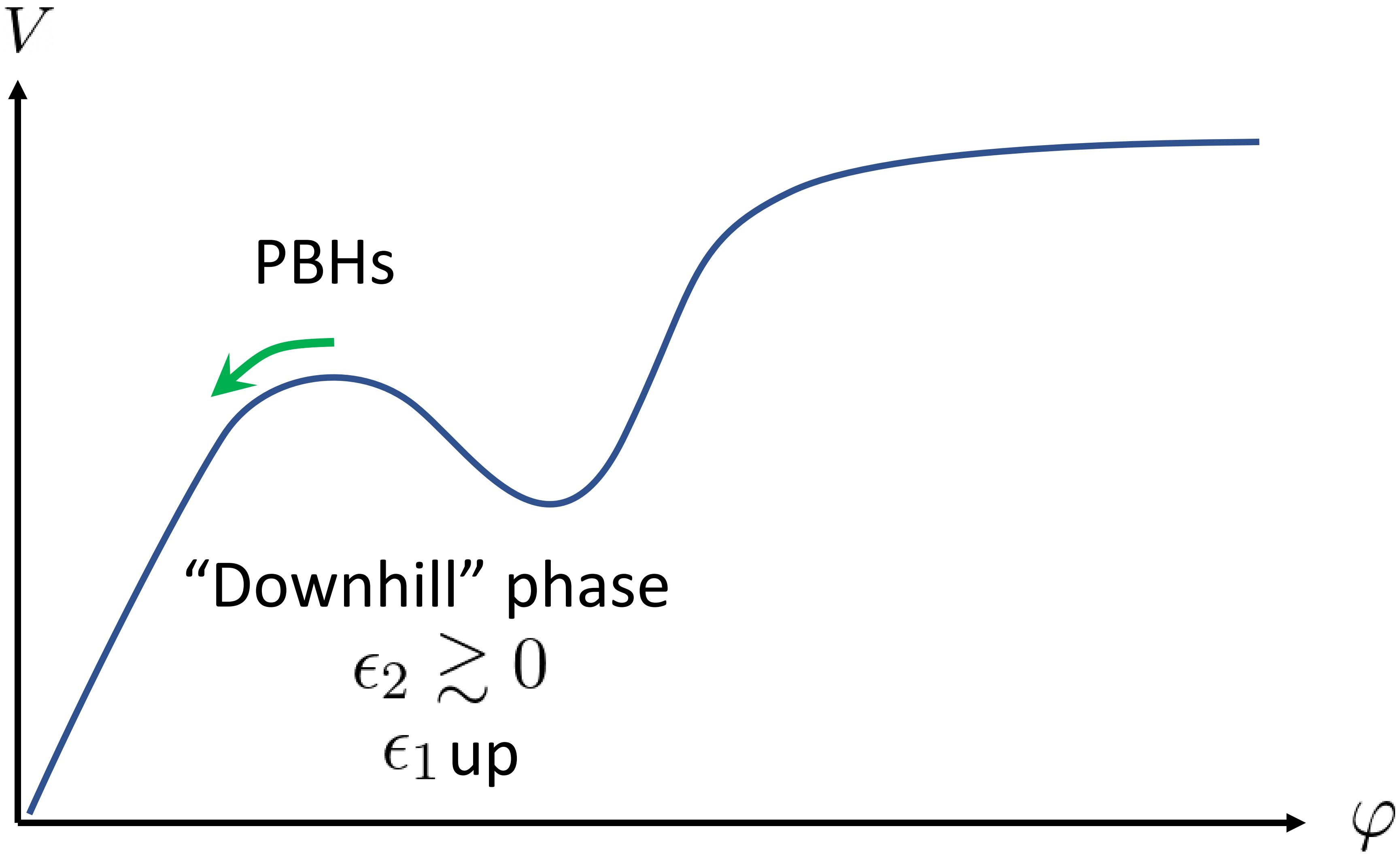


PBHs

“Uphill” phase

$\epsilon_2 \lesssim -6$

ϵ_1 down



PBHs

“Downhill” phase

$\epsilon_2 \gtrsim 0$

ϵ_1 up

Linear perturbations grow near feature

Comoving curvature perturbation $\mathcal{R} = \frac{\delta\varphi}{\sqrt{2\epsilon_1}}$

$$\ddot{\mathcal{R}}_k + H(3+\epsilon_2)\dot{\mathcal{R}}_k + \frac{k^2}{a^2}\mathcal{R}_k = 0$$

Linear perturbations grow near feature

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Vacuum initial conditions:

$$\mathcal{R}_k = \frac{1}{2a\sqrt{k\epsilon_1}} e^{ik/(aH)}$$

Late times:

$$\mathcal{R}_k \to \text{const. if } \ \epsilon_2 > -3$$

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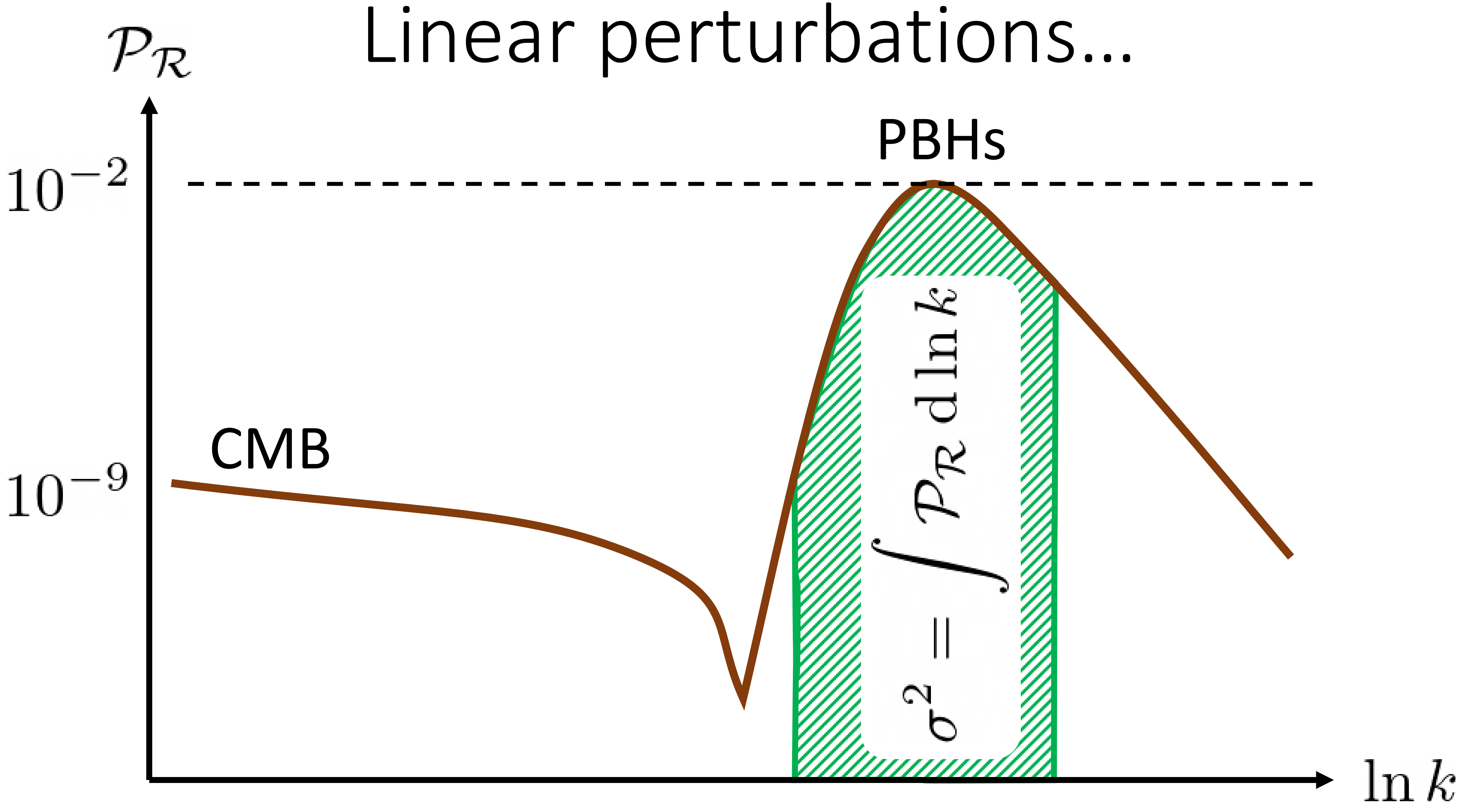
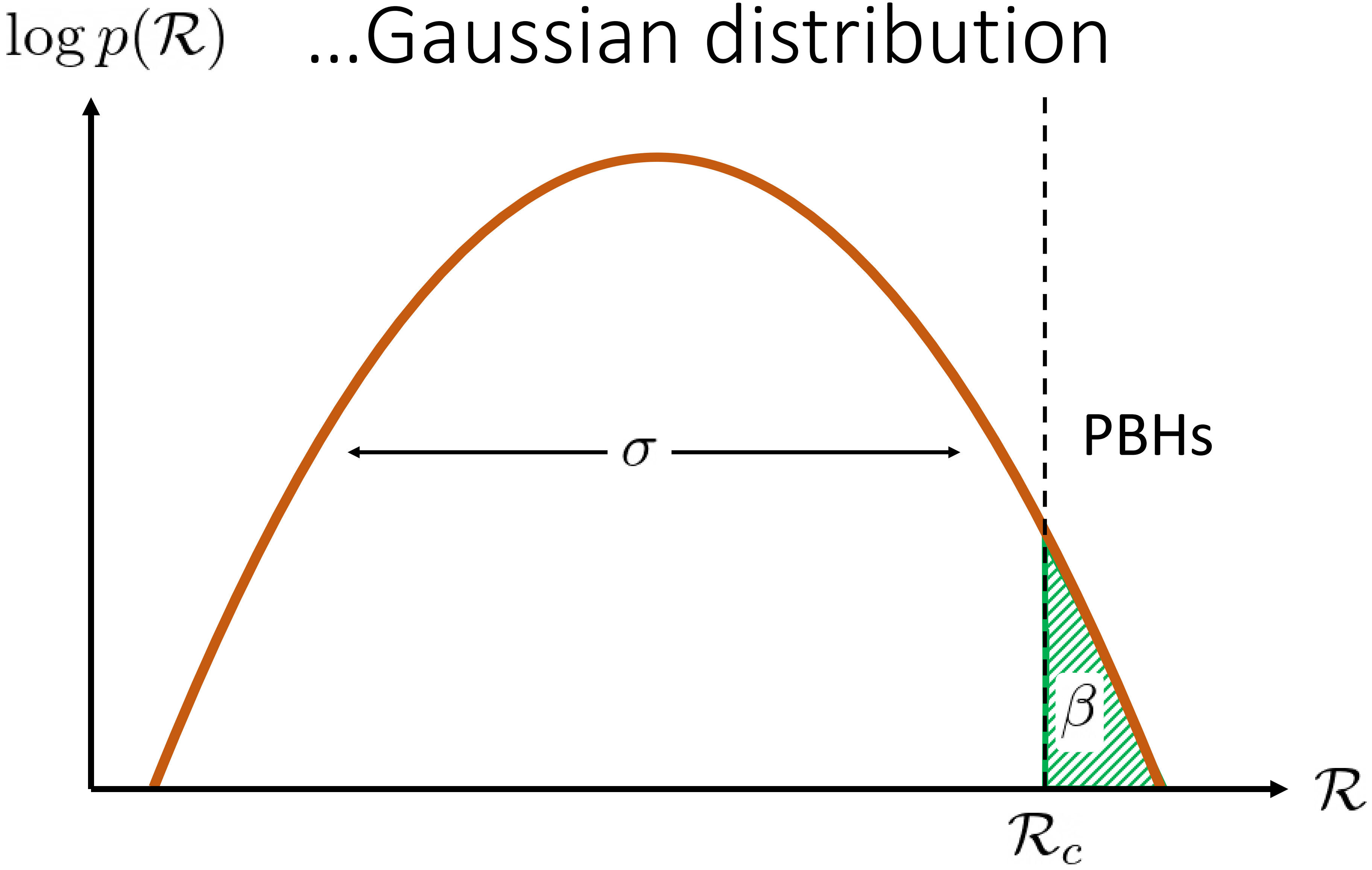
Define power spectrum: $\mathcal{P}_{\mathcal{R}}(k) \equiv \frac{k^3}{2\pi^2}|\mathcal{R}_k|^2$

Linear perturbations…



…Gaussian distribution



Why this picture is wrong

$\mathcal{R}$ is not the correct statistic for PBH formation

Perturbations in the tail are not Gaussian

I. (Semi-)inflection point inflation

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Approximations in two regimes

Sub-Hubble scales:

Linear perturbation theory good; neglect mode couplings

$$\delta\ddot{\varphi}_k + 3H\delta\dot{\varphi}_k + H^2\left(\frac{k^2}{a^2H^2} - \frac{3}{2}\epsilon_2 + \frac{1}{2}\epsilon_1\epsilon_2 - \frac{1}{4}\epsilon_2^2 - \frac{1}{2}\epsilon_2\epsilon_3\right)\delta\varphi_k = 0$$

Super-Hubble scales:

Local FLRW equations good; neglect gradient terms

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0$$

Approximations in two regimes

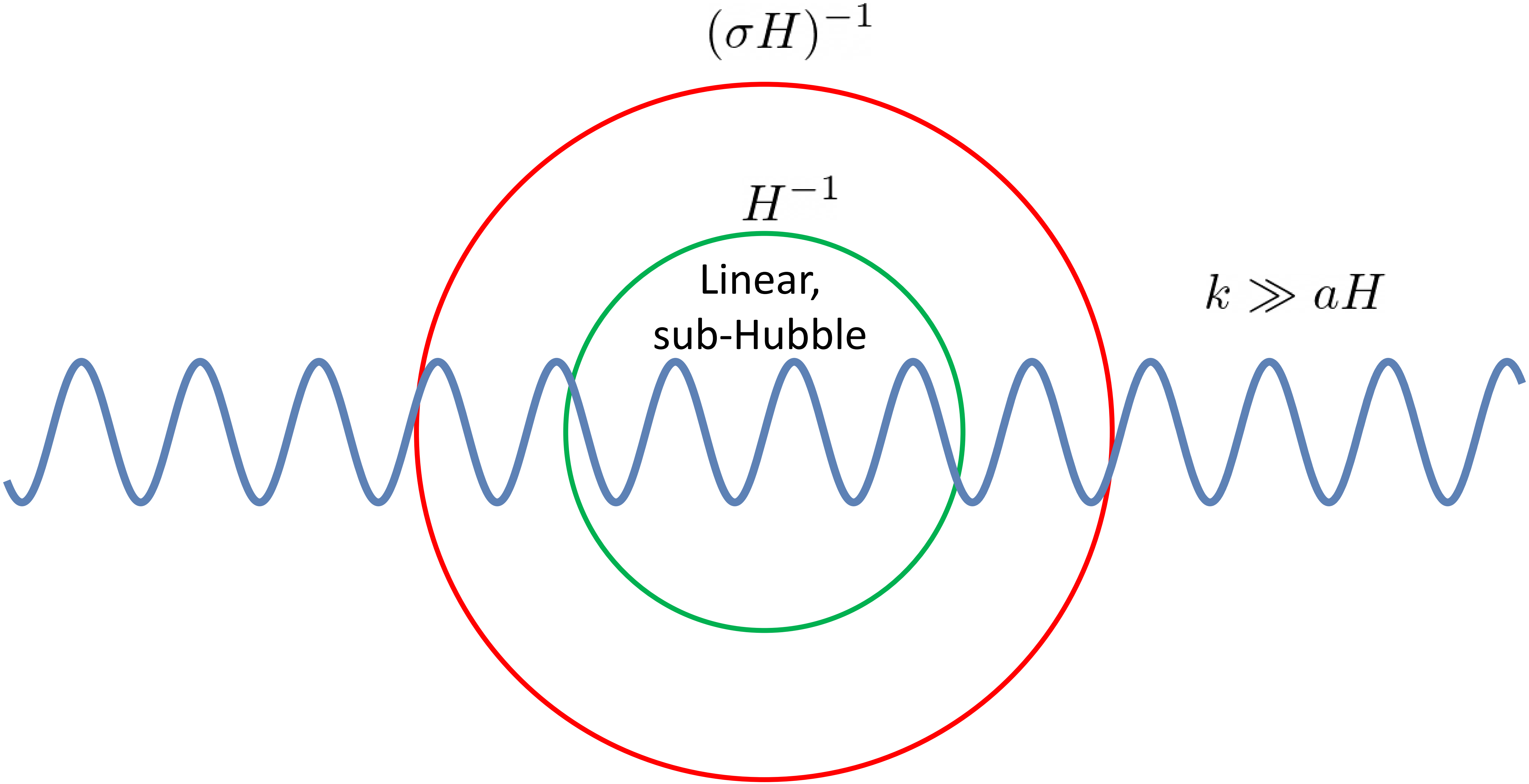
Inflaton field: $\varphi = \phi + \delta\phi$

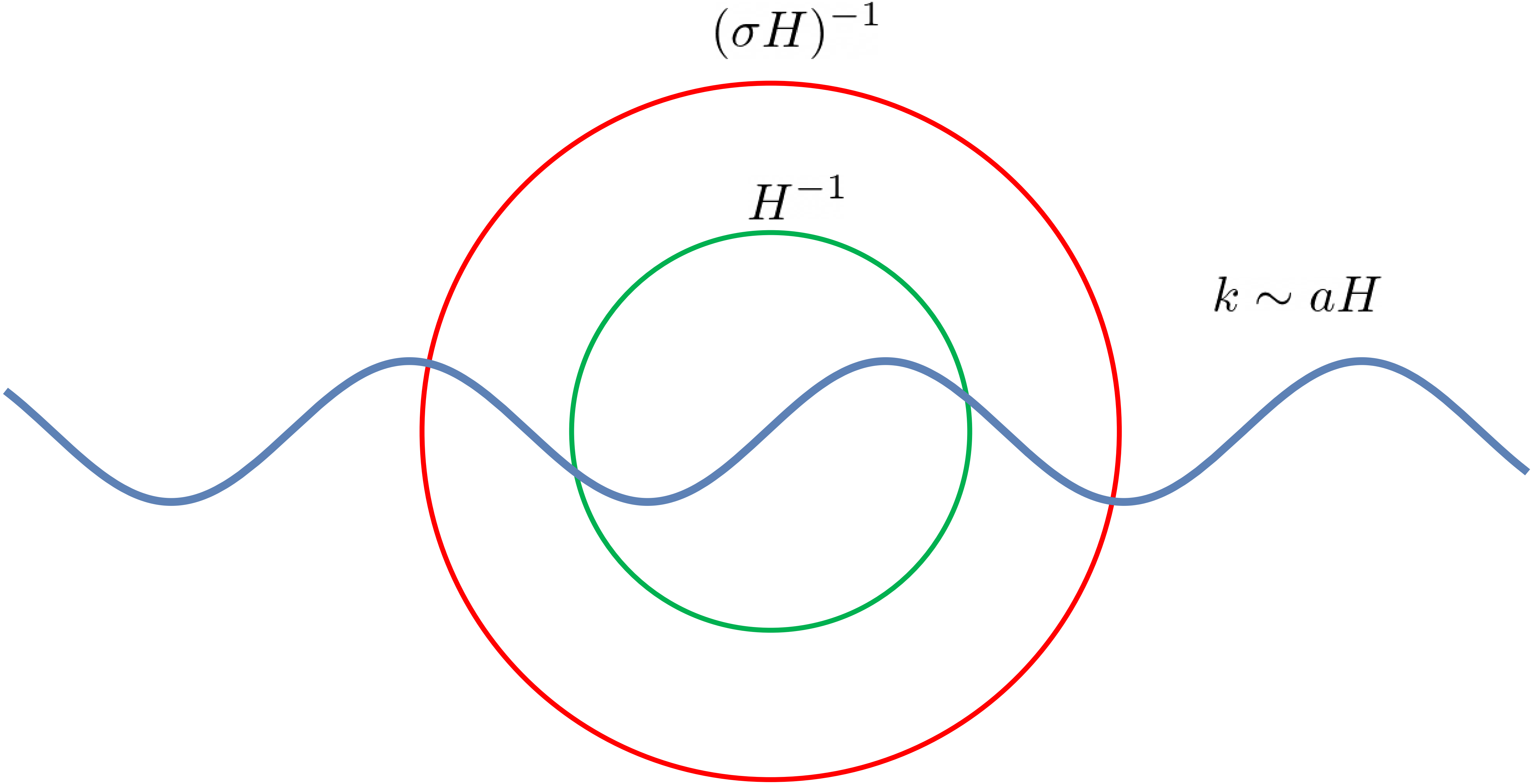
Coarse-grained:
FLRW

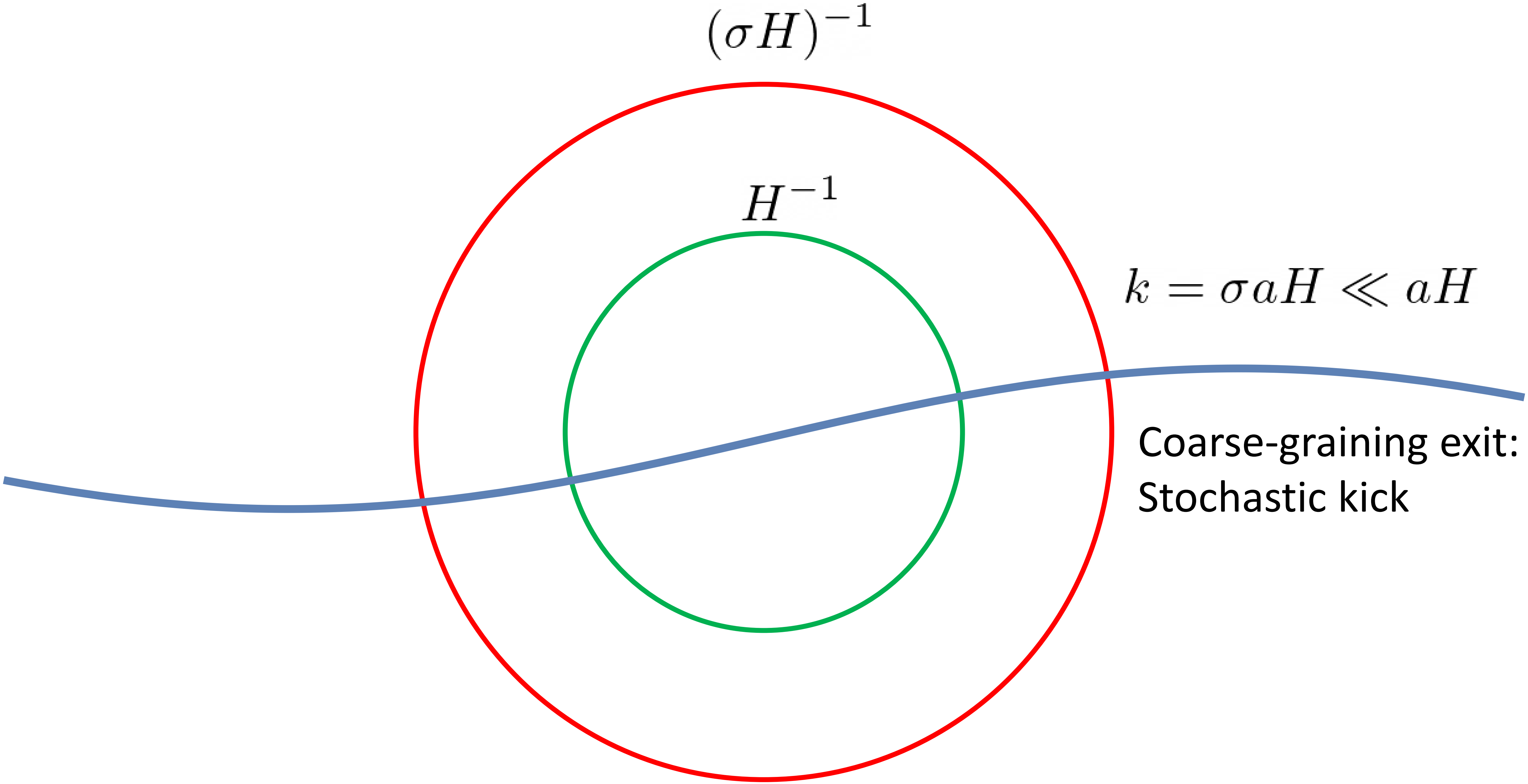
Short-wavelength:
linear perturbation theory

$$\phi \equiv \int_{k<k_\sigma} \frac{\mathrm{d}^3k}{(2\pi)^{2/3}} \varphi_k(N) e^{-i\vec{k}\cdot\vec{x}} \qquad \delta\phi \equiv \int_{k>k_\sigma} \frac{\mathrm{d}^3k}{(2\pi)^{2/3}} \varphi_k(N) e^{-i\vec{k}\cdot\vec{x}}$$

Patched together at the coarse-graining scale $k = k_\sigma \equiv \sigma a H$


$(\sigma H)^{-1}$
H^{-1}
Linear,
sub-Hubble
$k \gg aH$


$(\sigma H)^{-1}$
H^{-1}
$k \sim aH$



Coarse-graining exit:
Stochastic kick

Stochastic inflation

$$\phi' = \pi + \xi_\phi\,, \quad \pi' = -\left(3 - \frac{1}{2}\pi^2\right)\pi - \frac{V'(\phi)}{H^2} + \xi_\pi\,, \quad H^2 = \frac{V(\phi)}{3 - \frac{1}{2}\pi^2}$$

$$\delta\phi_k'' = -(3 - \frac{1}{2}\pi^2)\delta\phi_k' - \left[\frac{k^2}{a^2H^2} + \pi^2(3 - \frac{1}{2}\pi^2) + 2\pi\frac{V'(\phi)}{H^2} + \frac{V''(\phi)}{H^2}\right]\delta\phi_k$$

$$\langle\xi_\phi(N)\xi_\phi(N')\rangle = \frac{1}{6\pi^2}\frac{\mathrm{d}k_\sigma^3}{\mathrm{d}N}|\delta\phi_{k_\sigma}(N)|^2\delta(N - N')$$

$$\langle\xi_\pi(N)\xi_\pi(N')\rangle = \frac{1}{6\pi^2}\frac{\mathrm{d}k_\sigma^3}{\mathrm{d}N}|\delta\phi'_{k_\sigma}(N)|^2\delta(N - N')$$

$$\langle\xi_\phi(N)\xi_\pi(N')\rangle = \frac{1}{6\pi^2}\frac{\mathrm{d}k_\sigma^3}{\mathrm{d}N}\delta\phi_{k_\sigma}(N)\delta\phi'^*_{k_\sigma}(N)\delta(N - N')$$

$$\mathcal{R}_{<k} = \Delta N = N - \bar{N}$$

ΔN formalism

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)e^{2\zeta(x,t)}\mathrm{d}x^2$$

$$\Delta N \equiv N - \bar{N} = \mathcal{R} = \zeta$$

Stochastic ΔN formalism:

- solve stochastic system many times; include kicks up to scale k
- collect N on each run
- build statistics for coarse-grained curvature perturbation $\mathcal{R}_{<k}$

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$$\delta\phi_k'' = -(3-\frac{1}{2}\pi^2)\delta\phi_k' - \left[\frac{k^2}{a^2H^2} + \pi^2(3-\frac{1}{2}\pi^2) + 2\pi\frac{V'(\phi)}{H^2} + \frac{V''(\phi)}{H^2}\right]\delta\phi_k$$

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$$\mathcal{R}_{<k} = \Delta N = N - \bar{N}$$

Stochastic inflation

$$\phi' = \pi + \xi_\phi\,, \quad \pi' = -\left(3-\frac{1}{2}\pi^2\right)\pi - V'$$ $$H^2 = \frac{V(\phi)}{3-\frac{1}{2}\pi^2}$$

$$\delta\phi_k'' = -(3-\frac{1}{2}\pi^2)\delta\phi_k' - \Bigg[\frac{k^2}{}$$ $$) + 2\pi\frac{V'(\phi)}{H^2} + \frac{V''(\phi)}{H^2}\Bigg]\delta\phi_k$$

$$\langle\xi_\phi(N)\xi_\phi(N')\rangle = \frac{1}{}$$ $$\delta(N-N')$$

$$\langle\xi_\pi(N)\xi_\pi(N'$$ $$_{k_\sigma}(N)|^2\delta(N-N')$$

$$\langle\xi_\phi(N)\xi_\pi(N$$ $$\frac{k_\sigma}{\mathrm{d}N}\delta\phi_{k_\sigma}(N)\delta\phi'^*_{k_\sigma}(N)\delta(N-N')$$

$$\mathcal{R}_{<k} = \Delta N = N - \bar{N}$$

COMPLICATED

How to move forward?

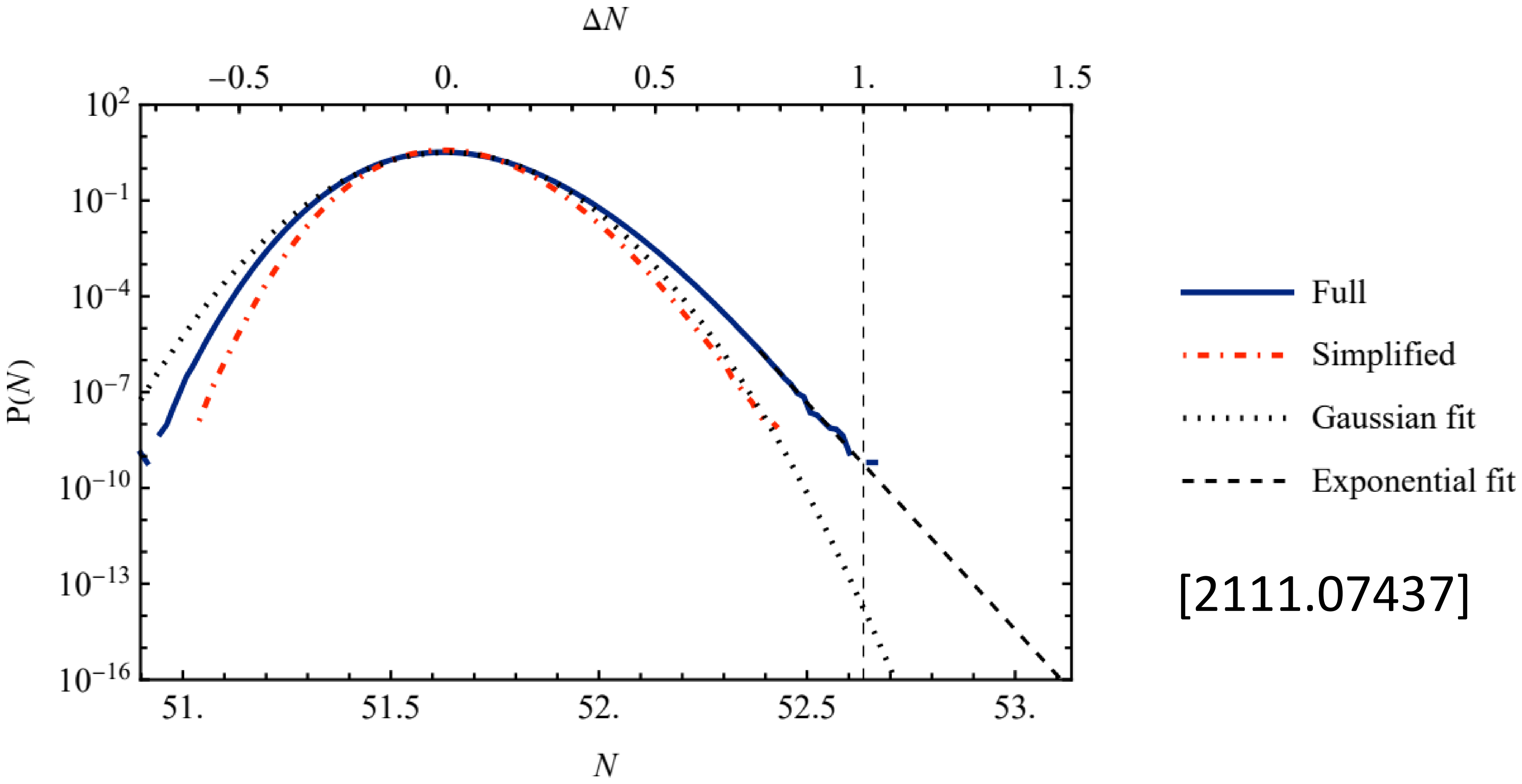
Analytical approximations?

$$\langle \xi_\phi(N)\xi_\phi(N')\rangle \approx \frac{H^2}{4\pi^2}\delta(N-N')$$

Full numerical computations?

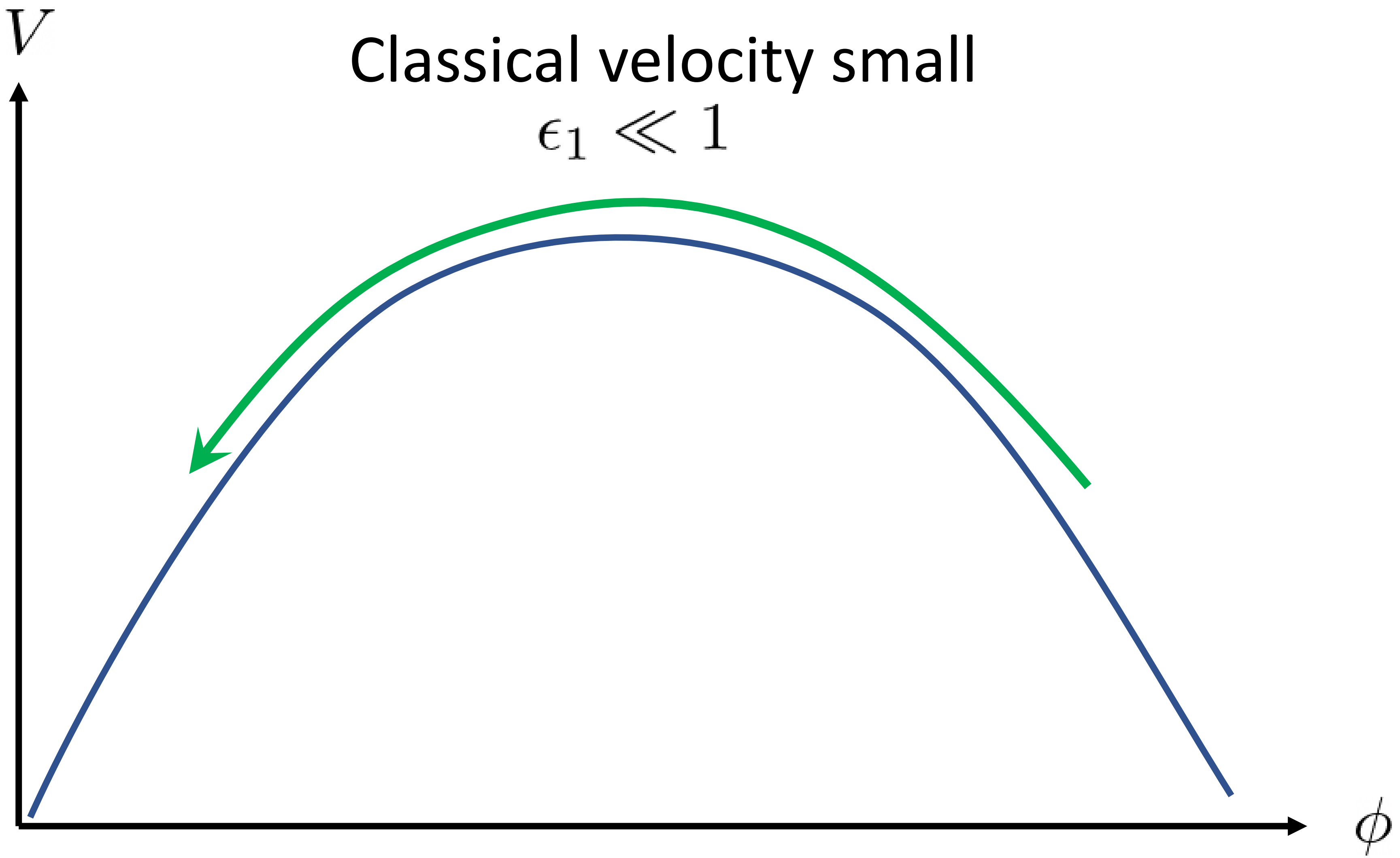
Full numerical computations

One million CPU hours



[2111.07437]

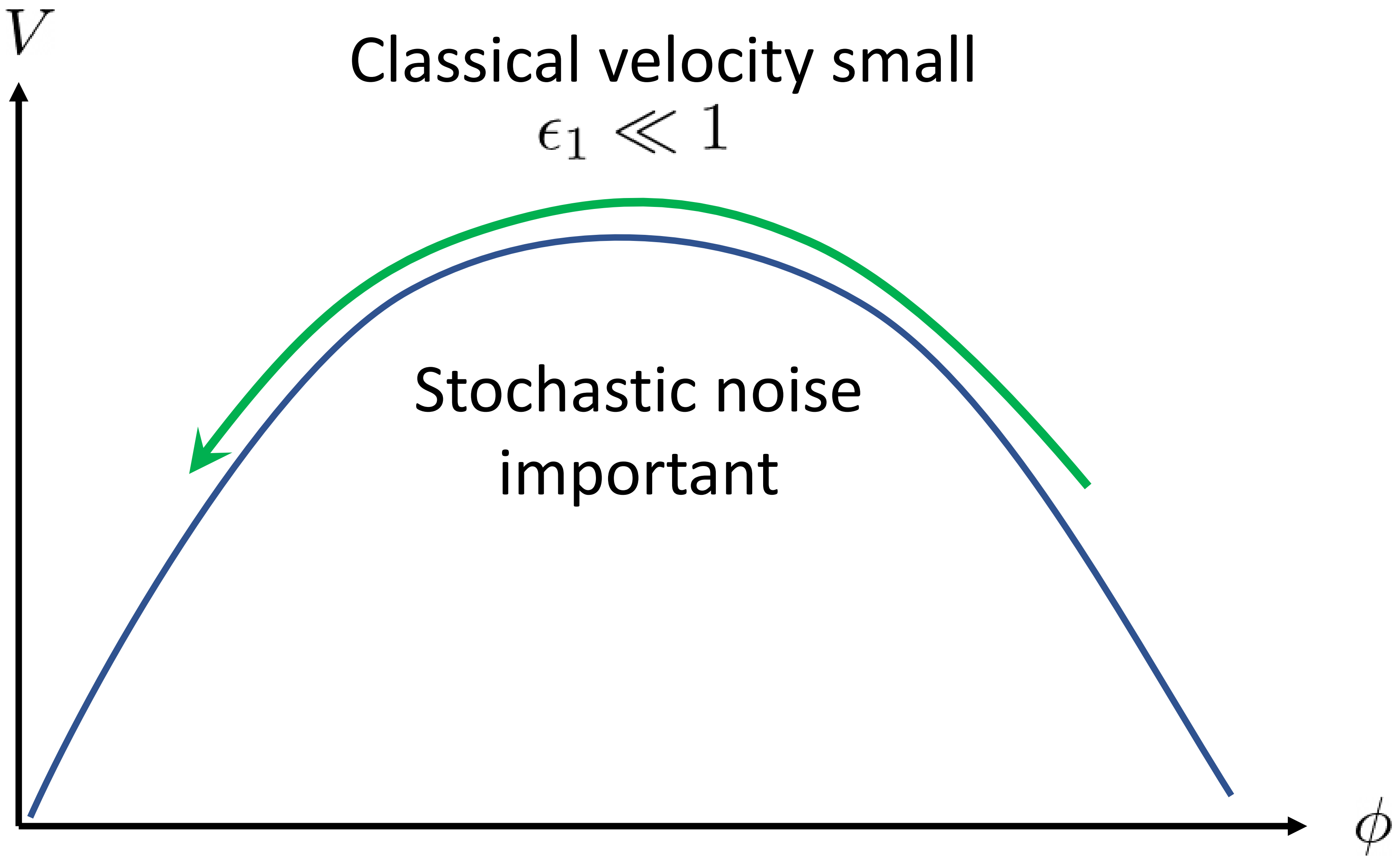
Zoom into the hilltop



Classical velocity small

$\epsilon_1 \ll 1$

Zoom into the hilltop

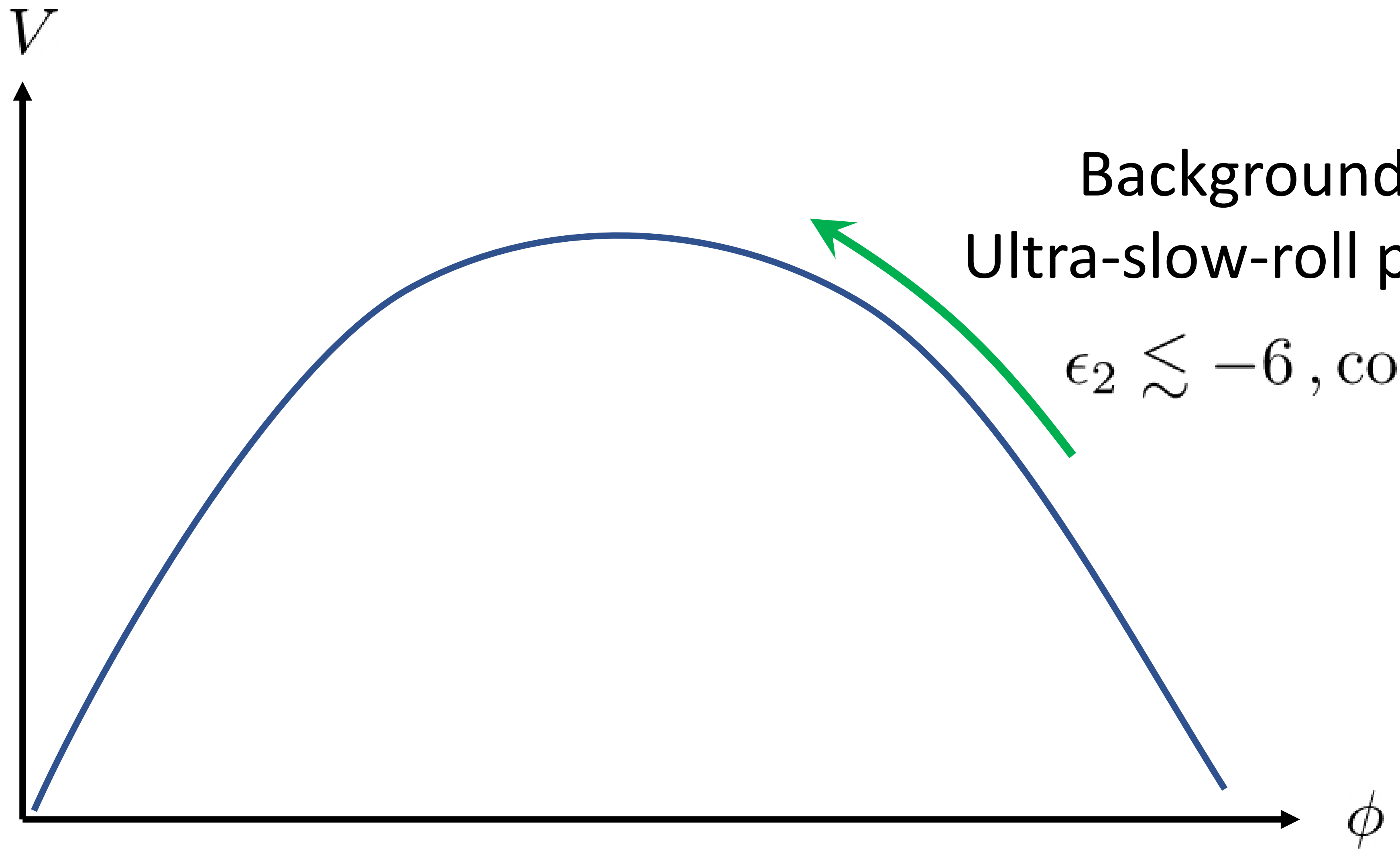


Classical velocity small

$\epsilon_1 \ll 1$

Stochastic noise important

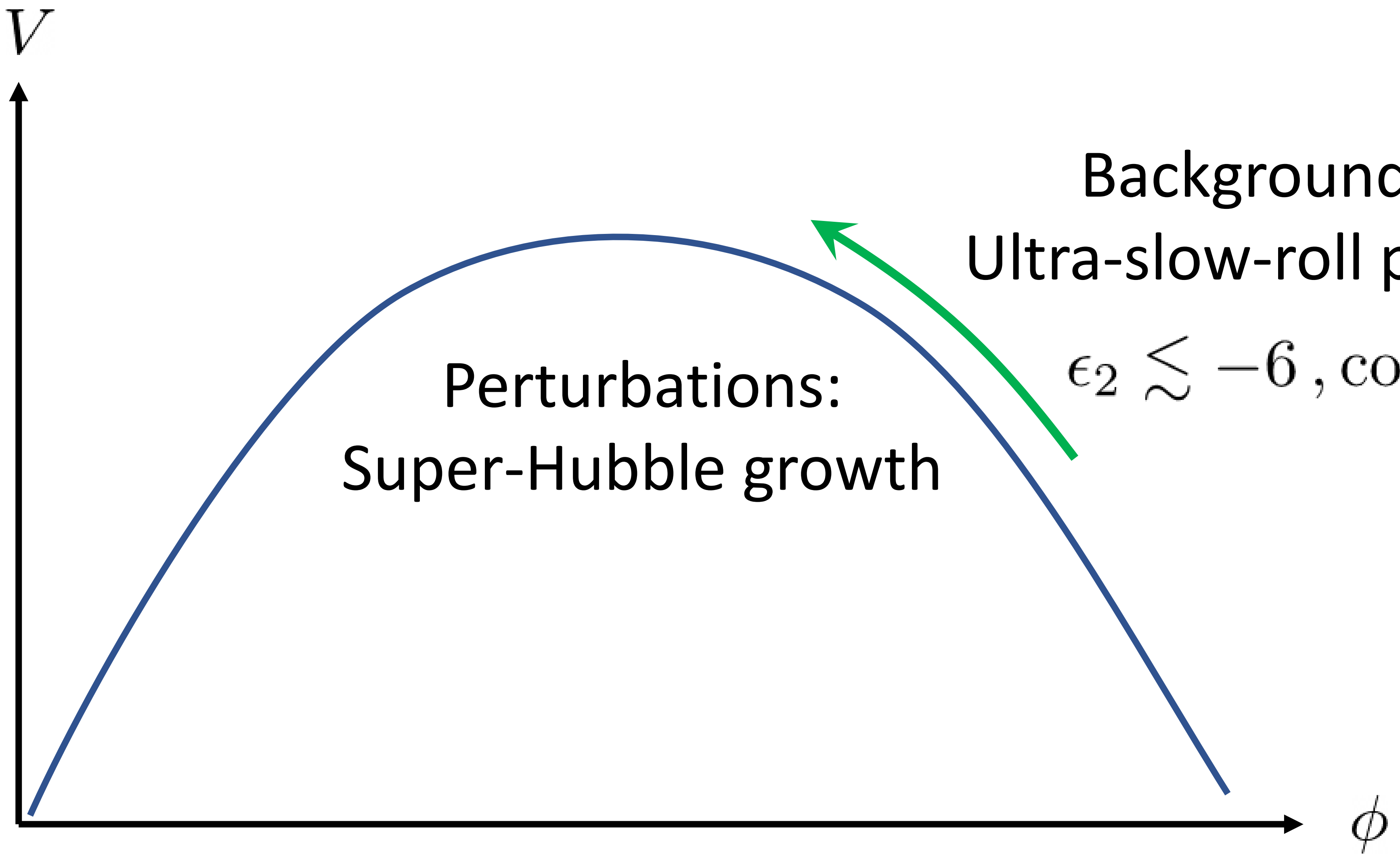
Zoom into the hilltop



Background:
Ultra-slow-roll phase

$\epsilon_2 \lesssim -6\,,\mathrm{const.}$

Zoom into the hilltop



Background:
Ultra-slow-roll phase

$$\epsilon_2 \lesssim -6\,, \mathrm{const.}$$

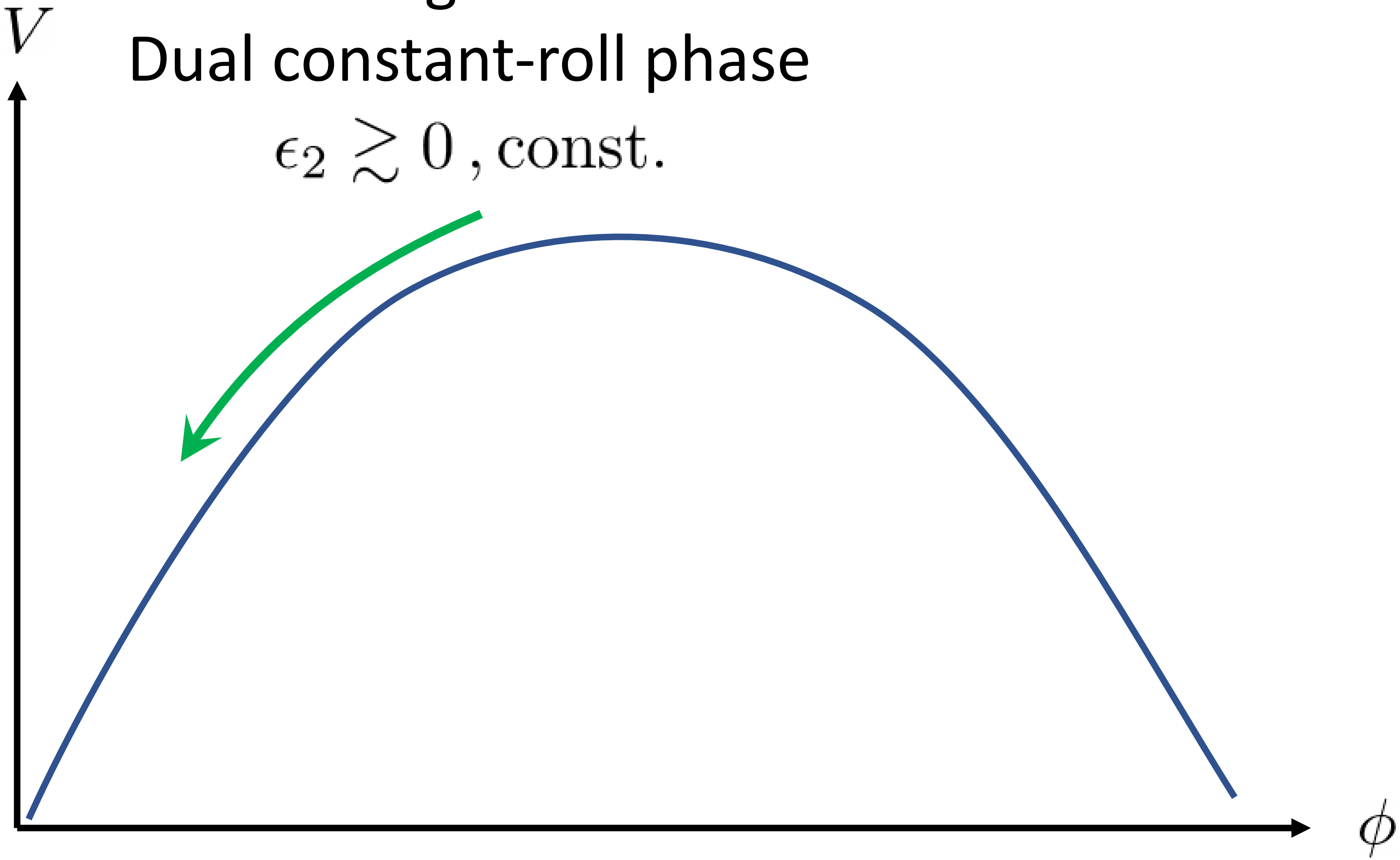
Perturbations:
Super-Hubble growth

Zoom into the hilltop

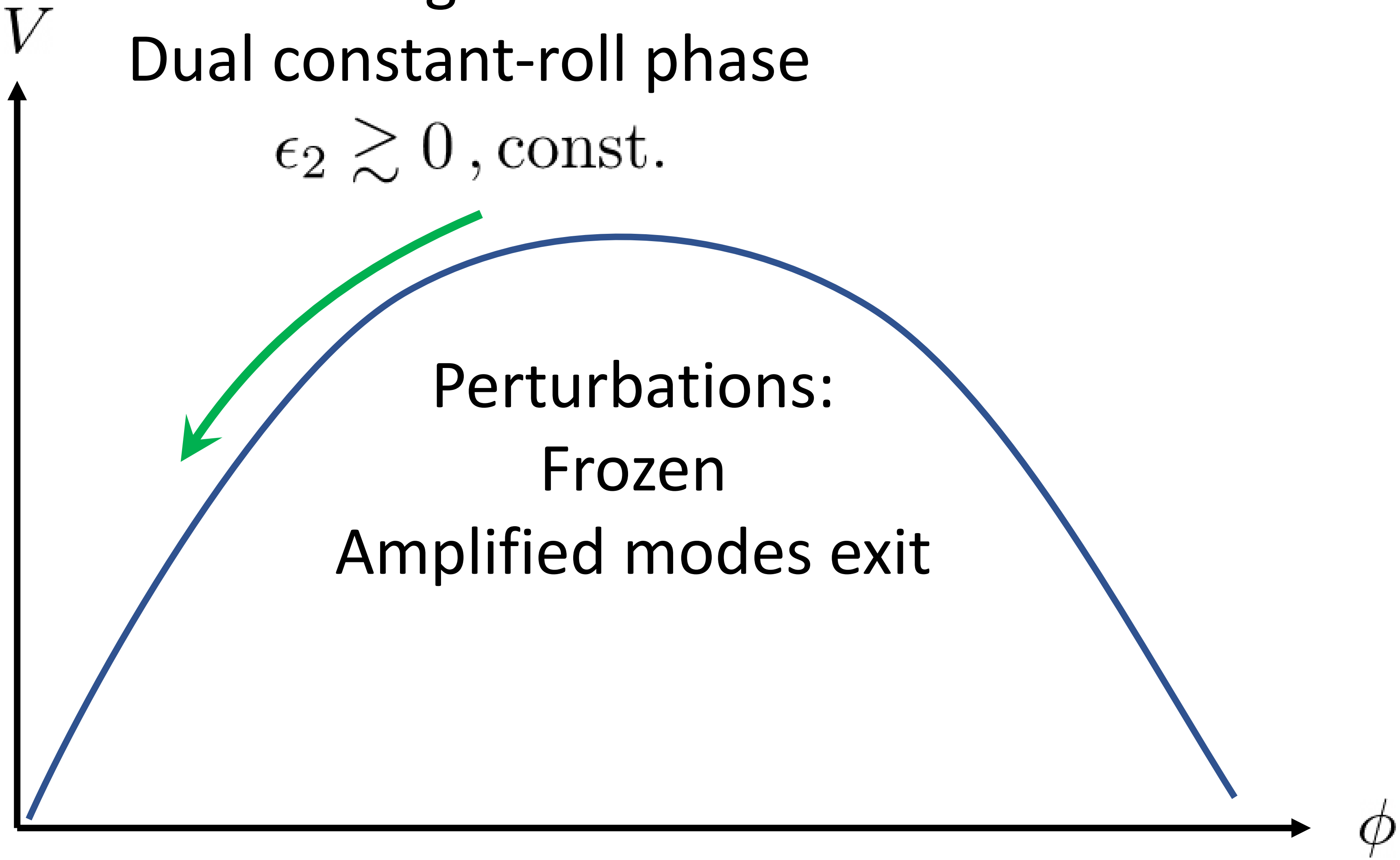
Background:

Dual constant-roll phase

$\epsilon_2 \gtrsim 0\,,\mathrm{const.}$



Zoom into the hilltop



Background:
Dual constant-roll phase
$\epsilon_2 \gtrsim 0\,,\mathrm{const.}$

Perturbations:
Frozen
Amplified modes exit

Equations simplify in dual constant-roll phase

Adiabatic perturbations:
motion along classical trajectory only

Noise independent of background stochasticity:
pre-compute power spectrum

Simplified stochastic equation:

$$\mathrm{d}\phi = \frac{\epsilon_2}{2}(\phi - \phi_0)\mathrm{d}N + \frac{\epsilon_2}{2}\phi_0 e^{\frac{\epsilon_2}{2}N}\sqrt{\mathcal{P}_{\mathcal{R}}(k_\sigma)\mathrm{d}N}\,\hat{\xi}_N$$

$$\langle \hat{\xi}_N \hat{\xi}_{N'} \rangle = \delta_{NN'}$$

Simplified stochastic equation:

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$$\langle \hat{\xi}_N \hat{\xi}_{N'} \rangle = \delta_{NN'}$$

$$\phi(N) = \phi_0\left(1 - e^{\frac{\epsilon_2}{2}N}\right) + \frac{\epsilon_2}{2}\phi_0 e^{\frac{\epsilon_2}{2}N} X_{<k_\sigma}$$

$$X_{<k} \equiv \sum_{\tilde{k}=k_{\mathrm{ini}}}^{k} \sqrt{\mathcal{P}_{\mathcal{R}}(\tilde{k})\,\mathrm{d}\ln k}\,\hat{\xi}_{\tilde{k}}$$

ΔN distribution

$$p(X_{<k}) = \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{X_{<k}^2}{2\sigma_k^2}}\,, \quad \sigma_k^2 \equiv \int_{k_{\mathrm{ini}}}^{k} \mathcal{P}_{\mathcal{R}}(\tilde{k})\, \mathrm{d}\ln\tilde{k}$$

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$$X_{<k} = \frac{2}{\epsilon_2}\left(1 - e^{-\frac{\epsilon_2}{2}\Delta N_{<k}}\right)$$

ΔN distribution

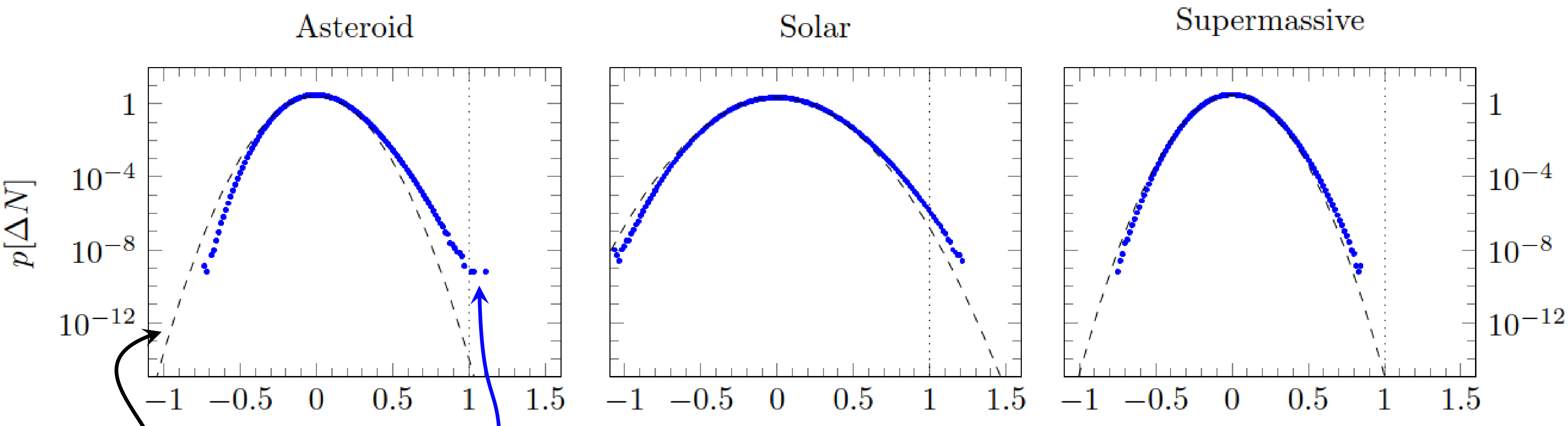
$$p(X_{<k}) = \frac{1}{\sqrt{2\pi\sigma_k}} e^{-\frac{X^2_{<k}}{2\sigma_k^2}}, \qquad \sigma_k^2 \equiv \int_{k_{\rm ini}}^{k} \mathcal{P}_\mathcal{R}(\tilde{k})\,\mathrm{d}\ln\tilde{k}$$

$$X_{<k} = \frac{2}{\epsilon_2}\left(1 - e^{-\frac{\epsilon_2}{2}\Delta N_{<k}}\right)$$

$$\boxed{p(\Delta N_{<k}) = \frac{1}{\sqrt{2\pi\sigma_k}} \exp\left[-\frac{2}{\sigma_k^2\epsilon_2^2}\left(1 - e^{-\frac{\epsilon_2}{2}\Delta N_{<k}}\right)^2 - \frac{\epsilon_2}{2}\Delta N_{<k}\right]}$$

$\Delta N_{<k} = \mathcal{R}_{<k}$

Comparison to numerics

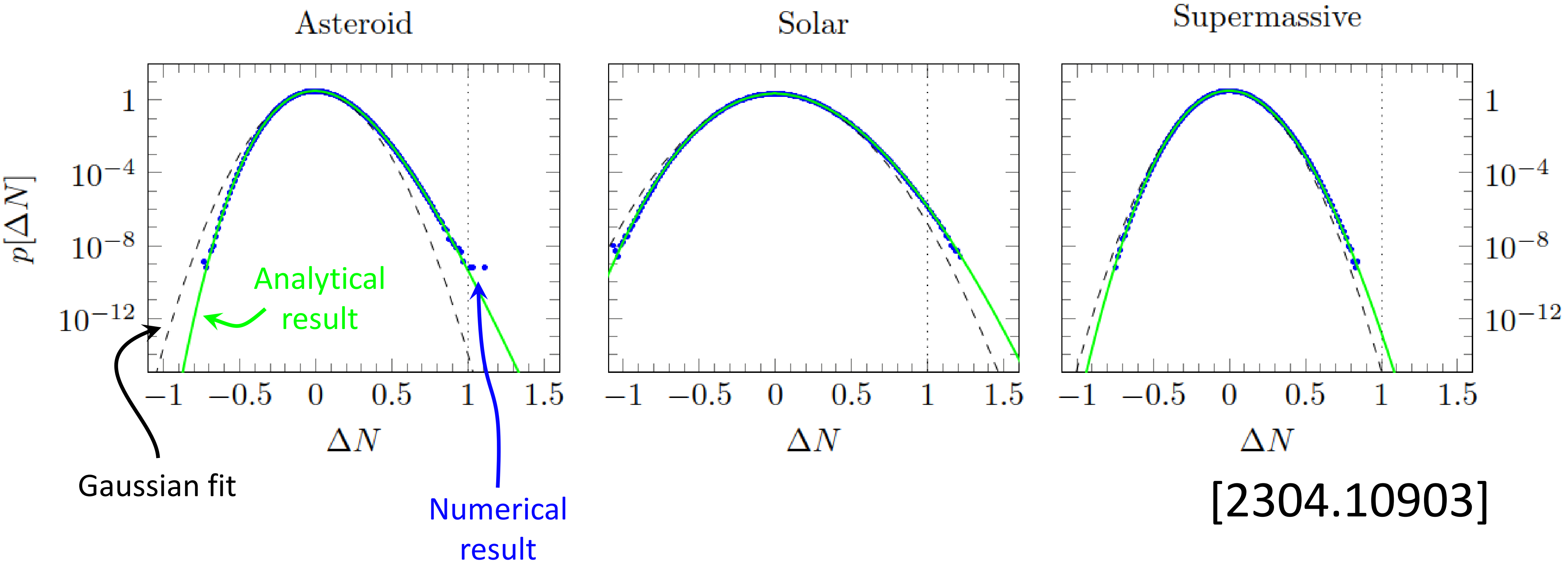


Gaussian fit

Numerical result

[2304.10903]

Comparison to numerics



[2304.10903]

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Compaction function: right tool for determining the collapse threshold

$$\mathcal{C} \equiv 2\frac{M_{\rm MS} - M_{\rm bg}}{R}$$

Collapse: $\mathcal{C}_{\rm max} > \mathcal{C}_c \approx 0.4$

Compaction function: right tool for determining the collapse threshold

$$\mathcal{C} \equiv 2\frac{M_{\rm MS} - M_{\rm bg}}{R}$$

Collapse: $\mathcal{C}_{\rm max} > \mathcal{C}_c \approx 0.4$

In inflationary variables:

$$\mathcal{C}(r) = \frac{2}{3}(1 - [1 + r\zeta'(r)]^2)$$

Assume spherical symmetry

$$r\zeta'(r) = \sum_k \frac{2k^2\,\mathrm{d}k}{\sqrt{2\pi}}\,\zeta_k\left[\cos(kr) - \frac{\sin(kr)}{kr}\right]$$

$$\zeta_k = \frac{\sqrt{2\pi}}{2k^3}\,\frac{\mathrm{d}\zeta_{<k}}{\mathrm{d}\ln k}$$

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$$r\zeta'(r) = \sum_k \frac{2k^2\,\mathrm{d}k}{\sqrt{2\pi}}\,\zeta_k \left[\cos(kr) - \frac{\sin(kr)}{kr}\right]$$

$$\zeta_k = \frac{\sqrt{2\pi}}{2k^3}\,\frac{\mathrm{d}\zeta_{<k}}{\mathrm{d}\ln k}$$

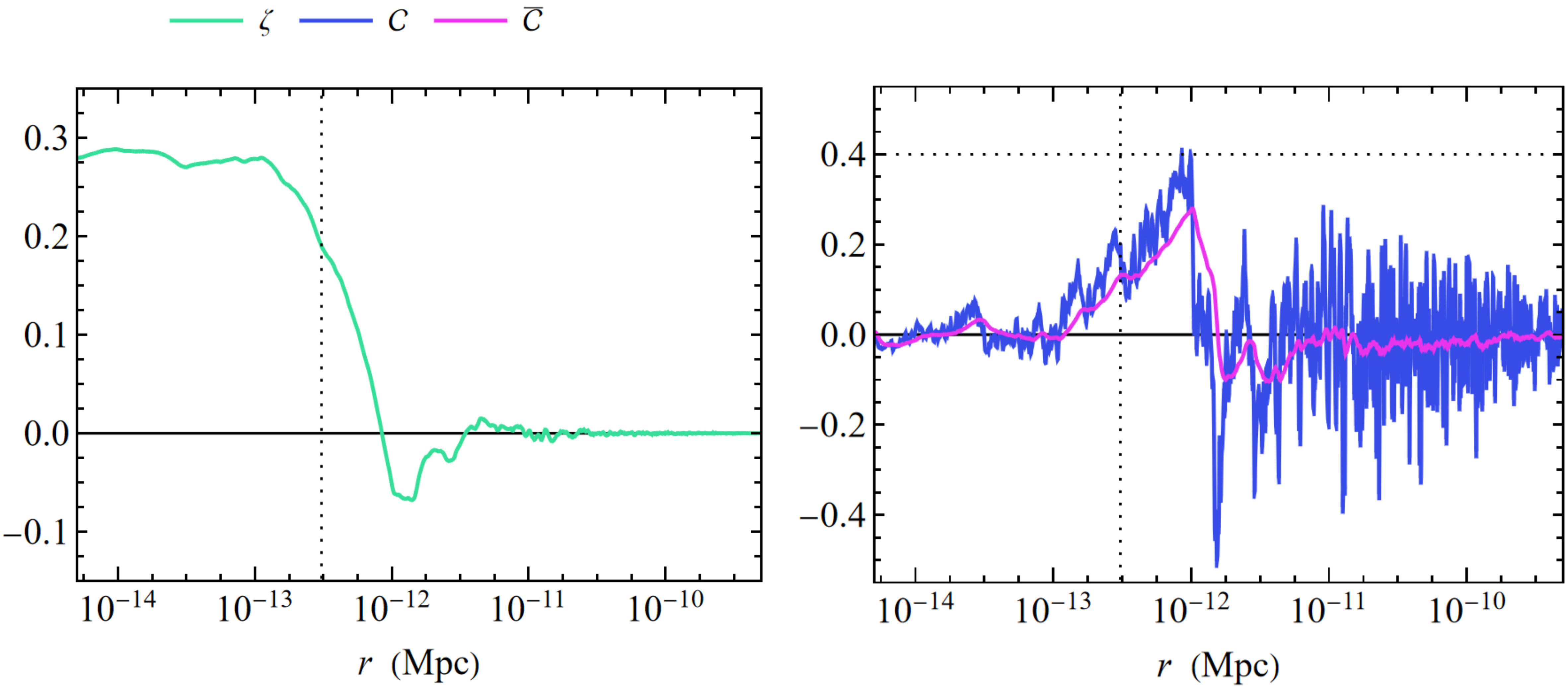
Recall: in the stochastic picture,

$$\zeta_{<k} = \Delta N_{<k} = -\frac{2}{\epsilon_2}\ln\left(1-\frac{\epsilon_2}{2}X_{<k}\right) = -\frac{2}{\epsilon_2}\ln\left(1-\frac{\epsilon_2}{2}\sum_{\tilde{k}=k_{\mathrm{ini}}}^{k}\sqrt{\mathcal{P}_{\mathcal{R}}(\tilde{k})\,\mathrm{d}\ln k}\;\hat{\xi}_{\tilde{k}}\right)$$

Vary k:
Full profile
in one patch of space!

Master formula

$$r\zeta'(r) = \sum_k \Bigg[-\frac{\hat{\xi}_k}{1-\frac{\epsilon_2}{2}X_{<k}}\sqrt{\mathcal{P}_\zeta(k)\,\mathrm{d}\ln k} + \frac{\epsilon_2}{4\left(1-\frac{\epsilon_2}{2}X_{<k}\right)^2}\mathcal{P}_\zeta(k)\,\mathrm{d}\ln k \Bigg] \times \left[\cos(kr) - \frac{\sin(kr)}{kr}\right]$$



ζ
C
$\overline{C}$
0.3
0.2
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0.0
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r (Mpc)
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−0.4
10^{-14}
10^{-13}
10^{-12}
10^{-11}
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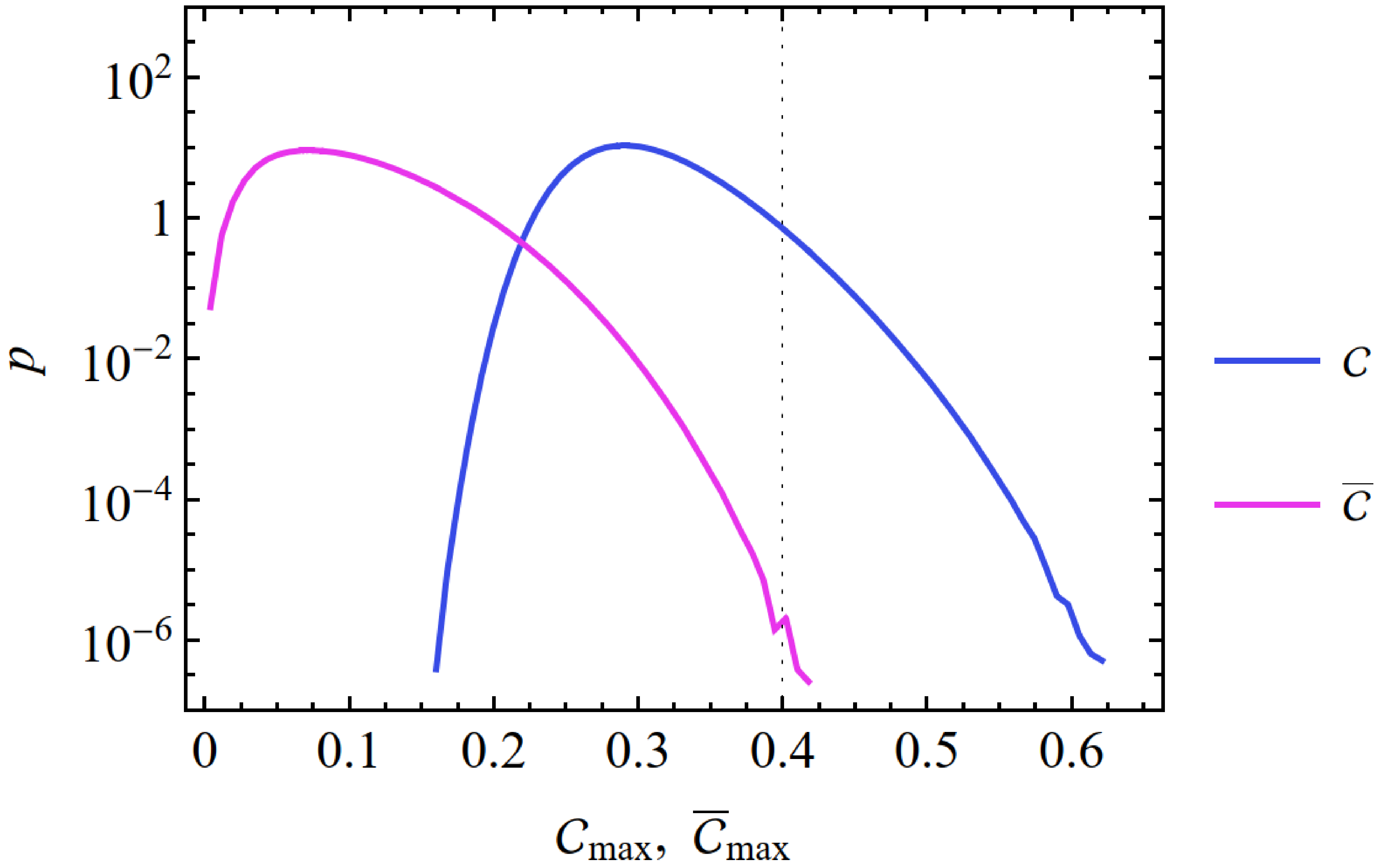
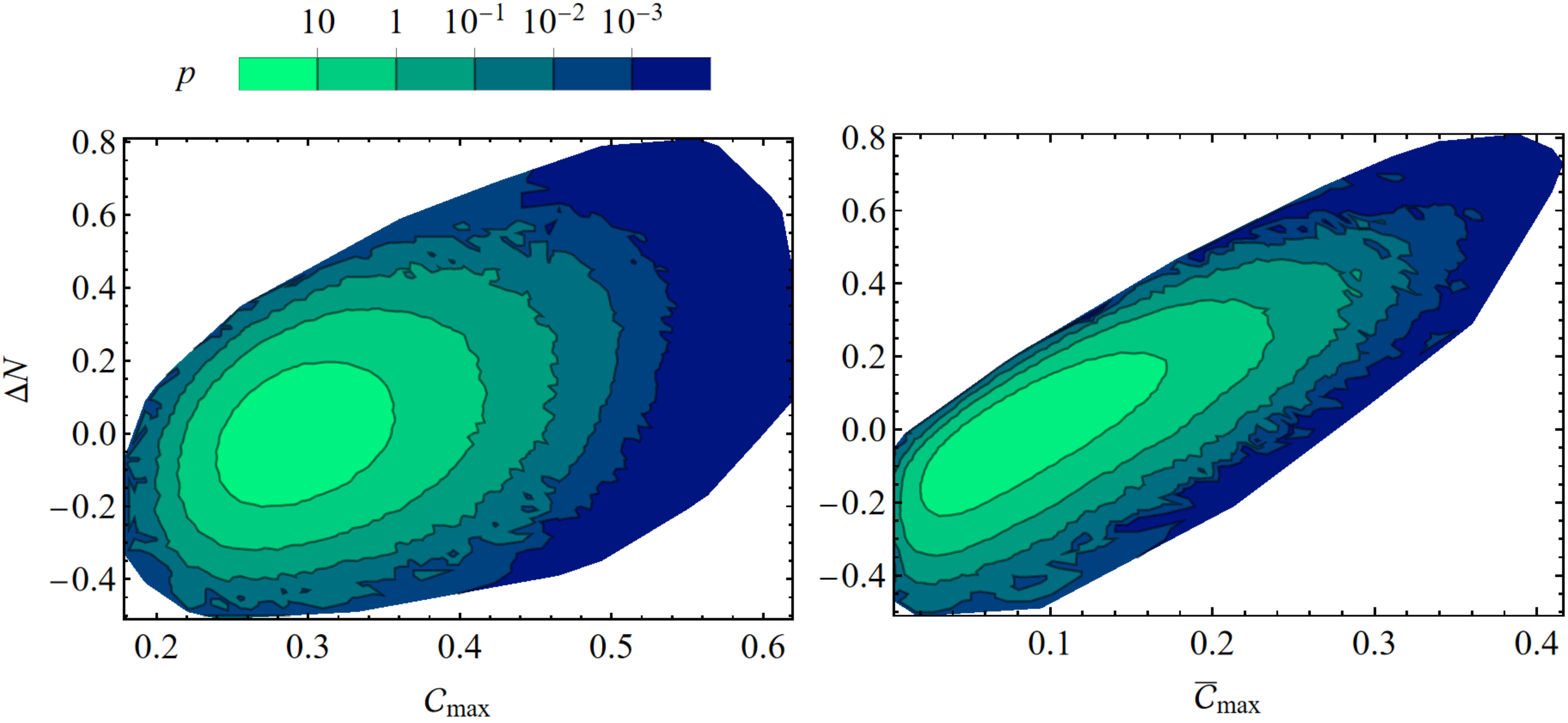
r (Mpc)


10^2
1
10^{-2}
10^{-4}
10^{-6}
p
0
0.1
0.2
0.3
0.4
0.5
0.6
$C_{\max},\ \overline{C}_{\max}$
C
$\overline{C}$



p
10
1
10^{-1}
10^{-2}
10^{-3}
ΔN
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
0.2
0.3
0.4
0.5
0.6
$C_{\max}$
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
0.1
0.2
0.3
0.4
$\overline{C}_{\max}$

Initial PBH fractions

Gaussian approximation, $\mathcal{R}_{<k} > 1$, fixed k: $\beta \approx 5 \times 10^{-16}$

Non-Gaussian statistics, $\mathcal{R}_{<k} > 1$, fixed k: $\beta \approx 2.2 \times 10^{-11}$

$\bar{\mathcal{C}}_{\max} > 0.4 : \quad \beta \approx 1.4 \times 10^{-8}$

$\mathcal{C}_{\max} > 0.4 : \quad \beta \approx 0.016$

I. (Semi-)inflection point inflation

II. Stochastic inflation

III. Black hole statistics

IV. An axion-curvaton model

Axion-like curvaton

$$V(\psi) = \Lambda_a^4 \left[1 - \cos\left(\frac{N_{\mathrm{DW}}\psi}{f_a}\right)\right]$$

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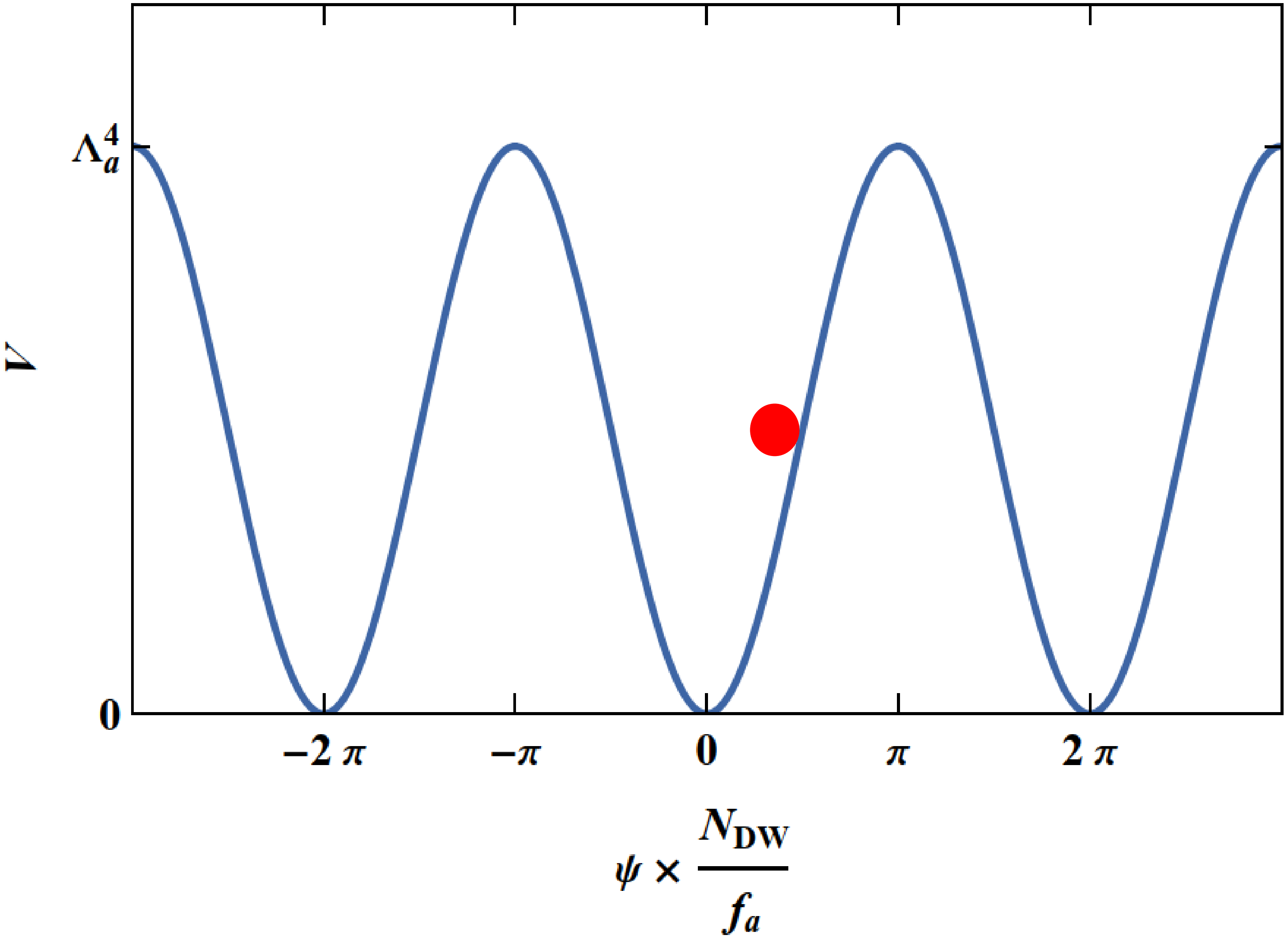
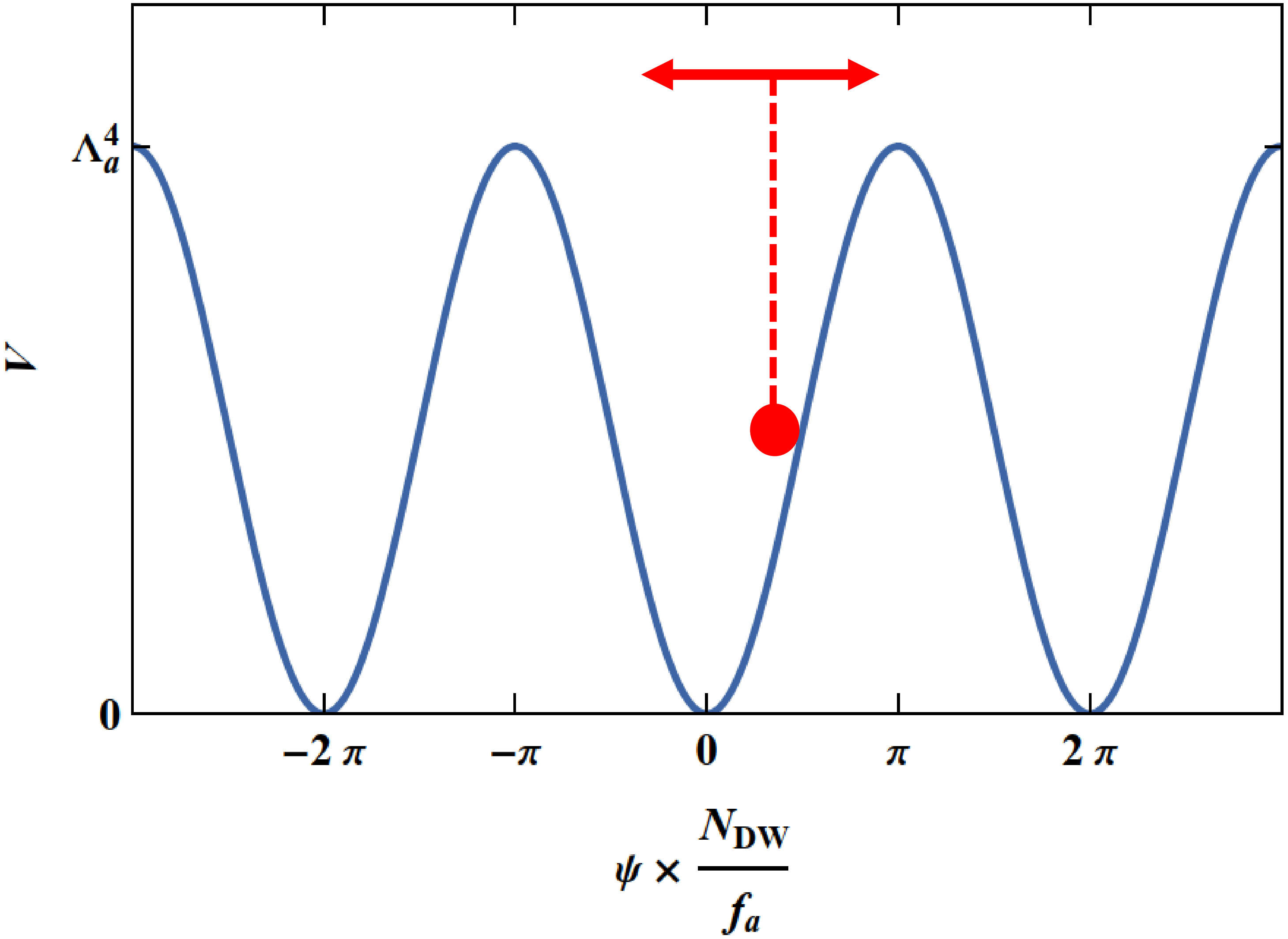
After inflation:

$$\psi'' + \left[\left(3 - \frac{1}{2}\psi'^2\right) - \frac{2}{3}\frac{\rho_r}{H^2}\right]\psi' + \frac{V'}{H^2} = 0\,, \quad H^2 = \frac{V + \rho_r}{3 - \frac{1}{2}\psi'^2}\,, \quad \rho_r = \rho_{\rm dec}e^{-4N_p}$$

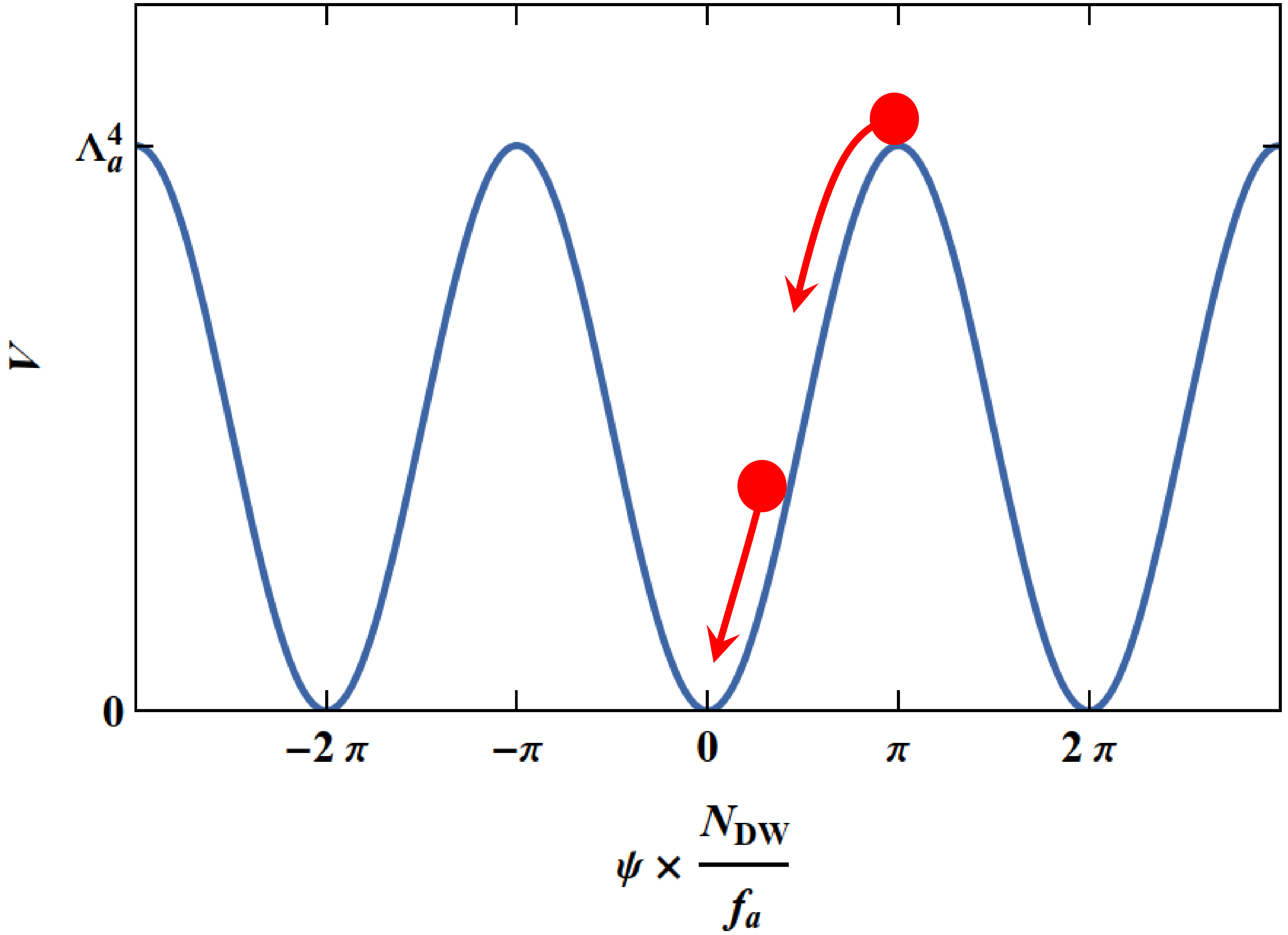
Axion-like curvaton



Axion-like curvaton



Axion-like curvaton

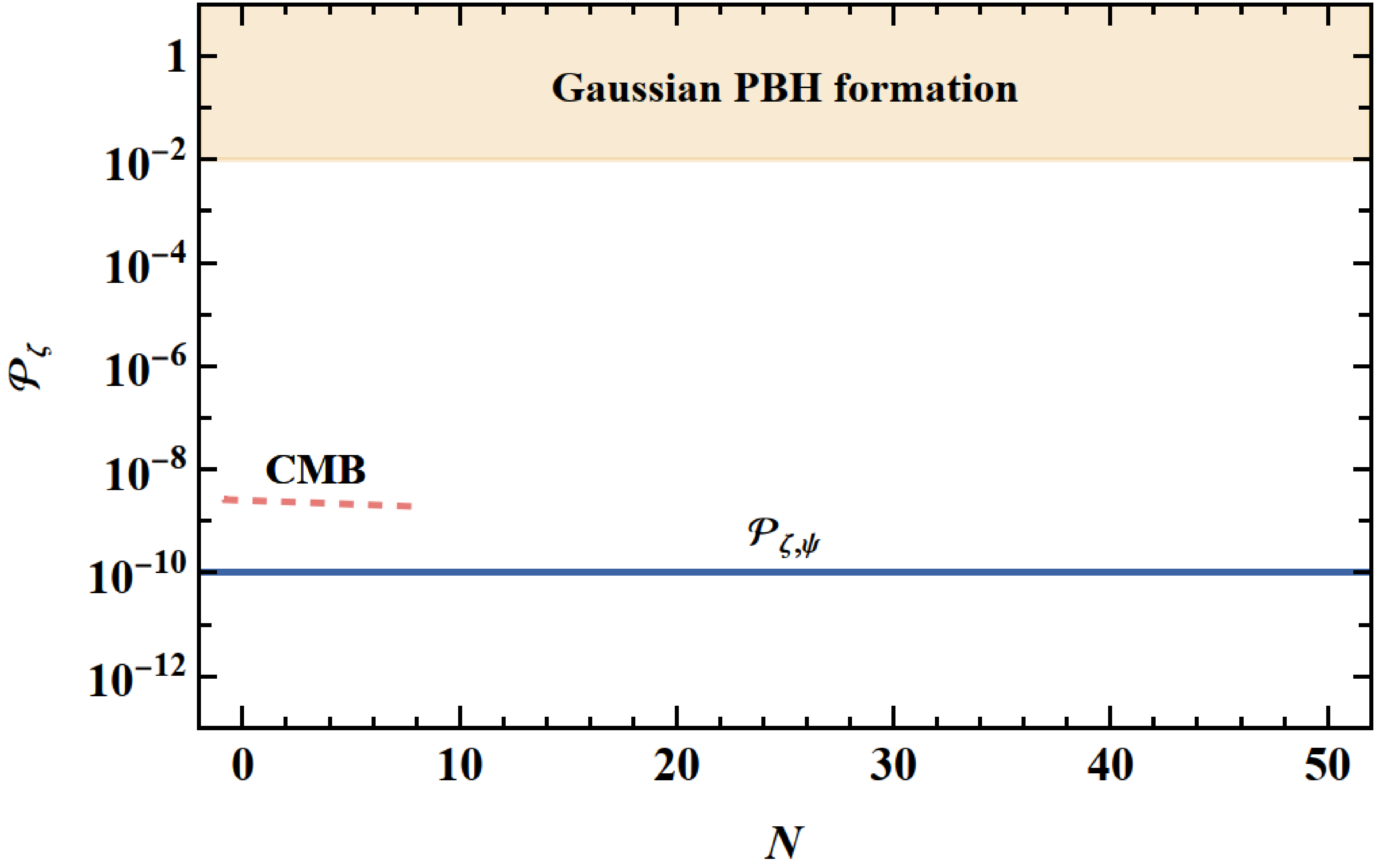


Cosmological perturbations

Curvaton decays;

curvature perturbation through ΔN formalism

$$\mathcal{P}_{\zeta,\psi}(k) = \mathcal{P}_{\psi}(k)\tilde{N}_{\psi_0}^2 = \frac{H^2(k)}{4\pi^2}\tilde{N}_{\psi_0}^2$$


Gaussian PBH formation
CMB
$\mathcal{P}_{\zeta,\psi}$
$\mathcal{P}_\zeta$
N
1
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
0
10
20
30
40
50

Cosmological perturbations

Curvaton decays;

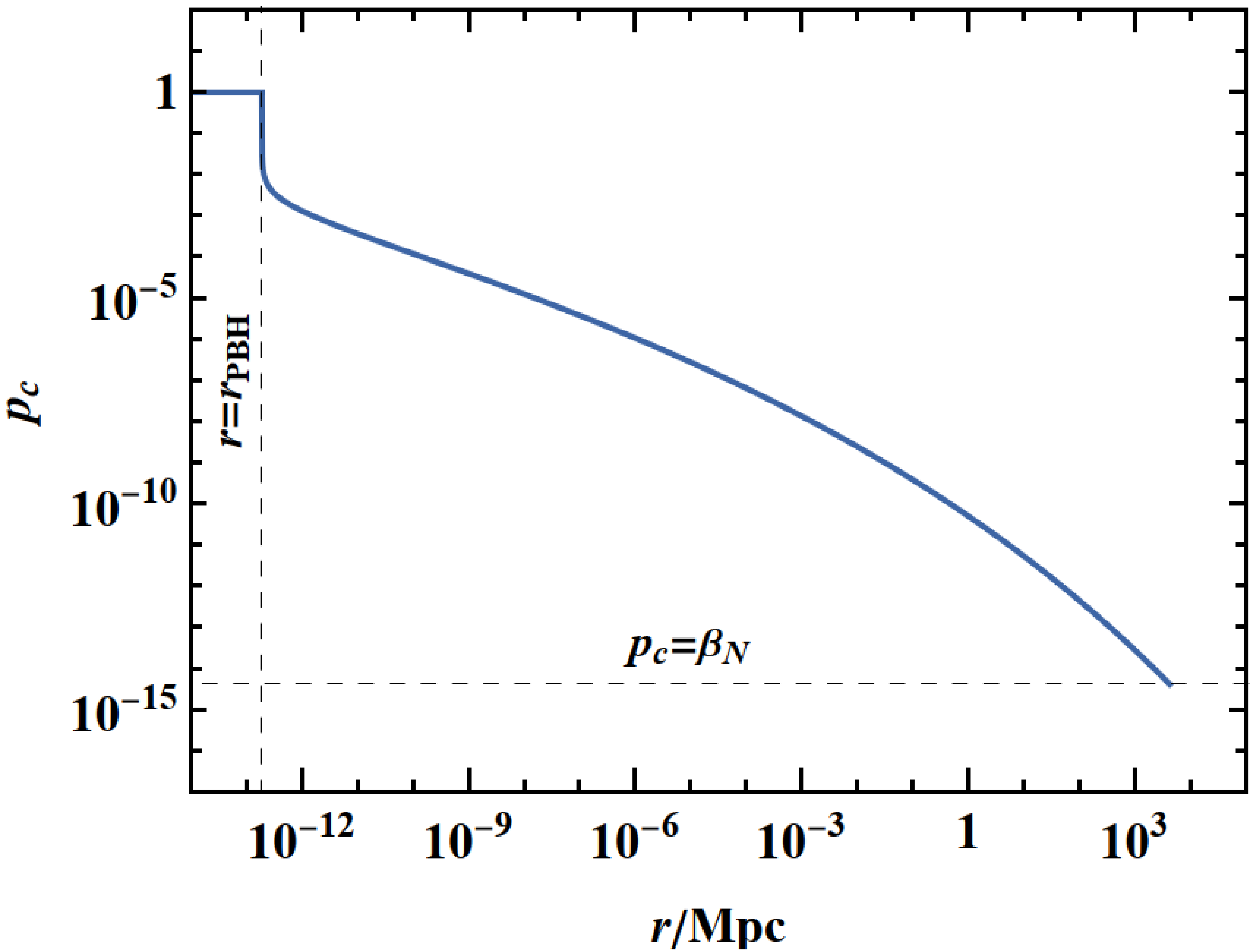
curvature perturbation through ΔN formalism

$\mathcal{C}_l$

$$\mathcal{C} = \frac{2}{3}\left[1-\left(1+\frac{\mathrm{d}\zeta(r)}{\mathrm{d}\ln r}\right)^2\right]$$

$$P(\mathcal{C}_l, N) = \frac{b}{2\sqrt{N\pi}\mathcal{C}_l^2} e^{-\frac{(\pi-\theta_0)^2}{2q^2 N}-\frac{1}{2}}$$

Clustering of PBHs

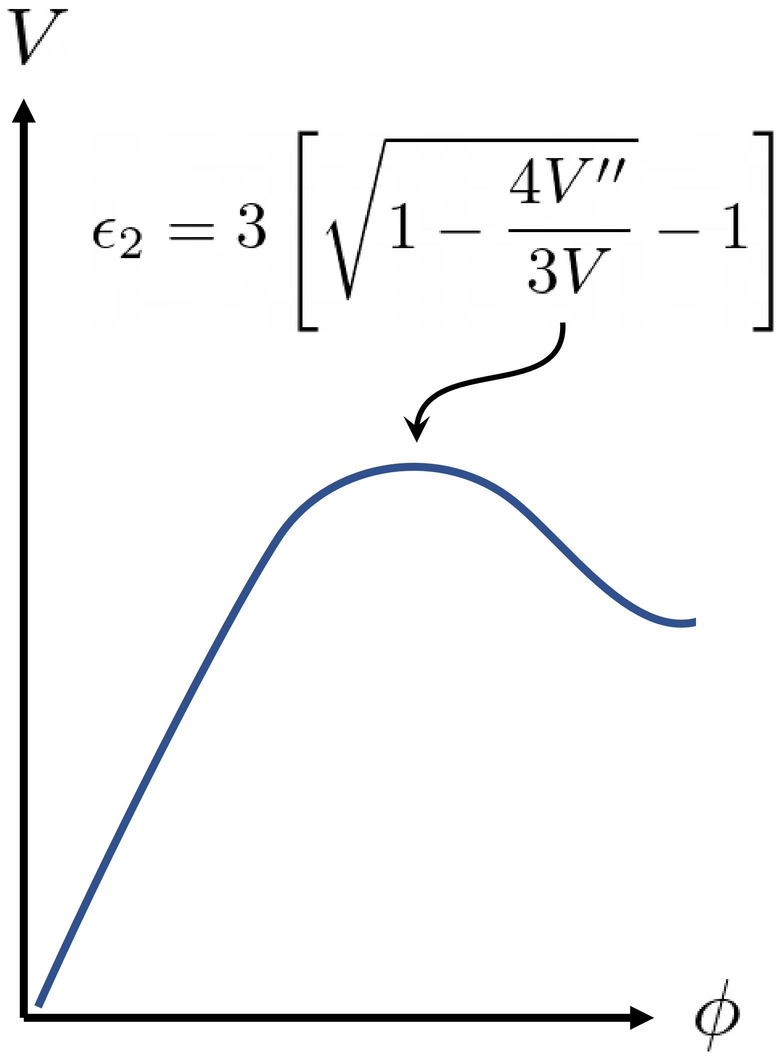


Conclusions

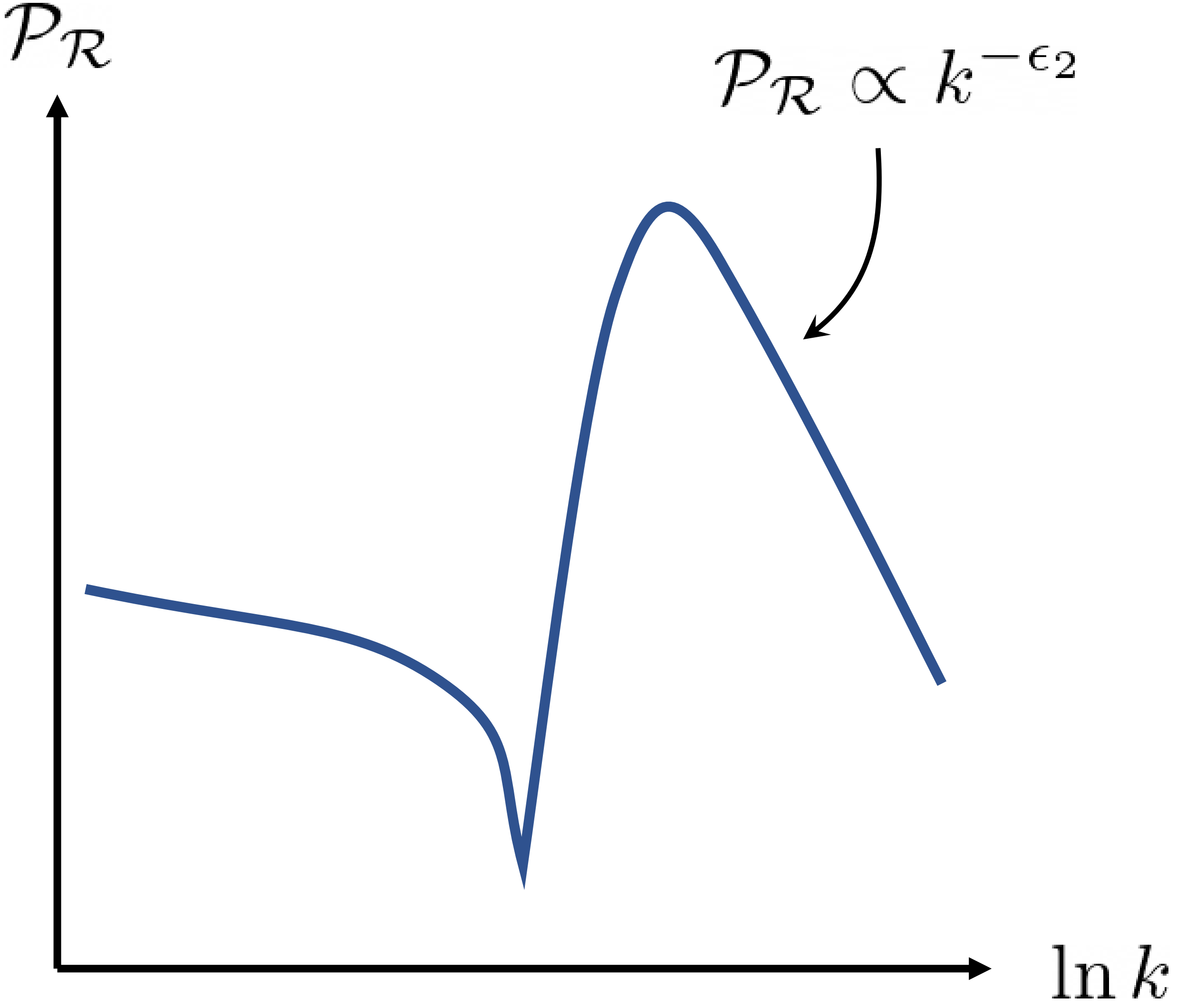
Stochastic inflation introduces non-Gaussian corrections to PBH statistics

Compaction function formalism needed for accurate results

Future directions:
resolving sharp peaks, considering PBH clustering



$$\epsilon_2 = 3\left[\sqrt{1 - \frac{4V''}{3V}} - 1\right]$$



$$\mathcal{P}_{\mathcal{R}} \propto k^{-\epsilon_2}$$

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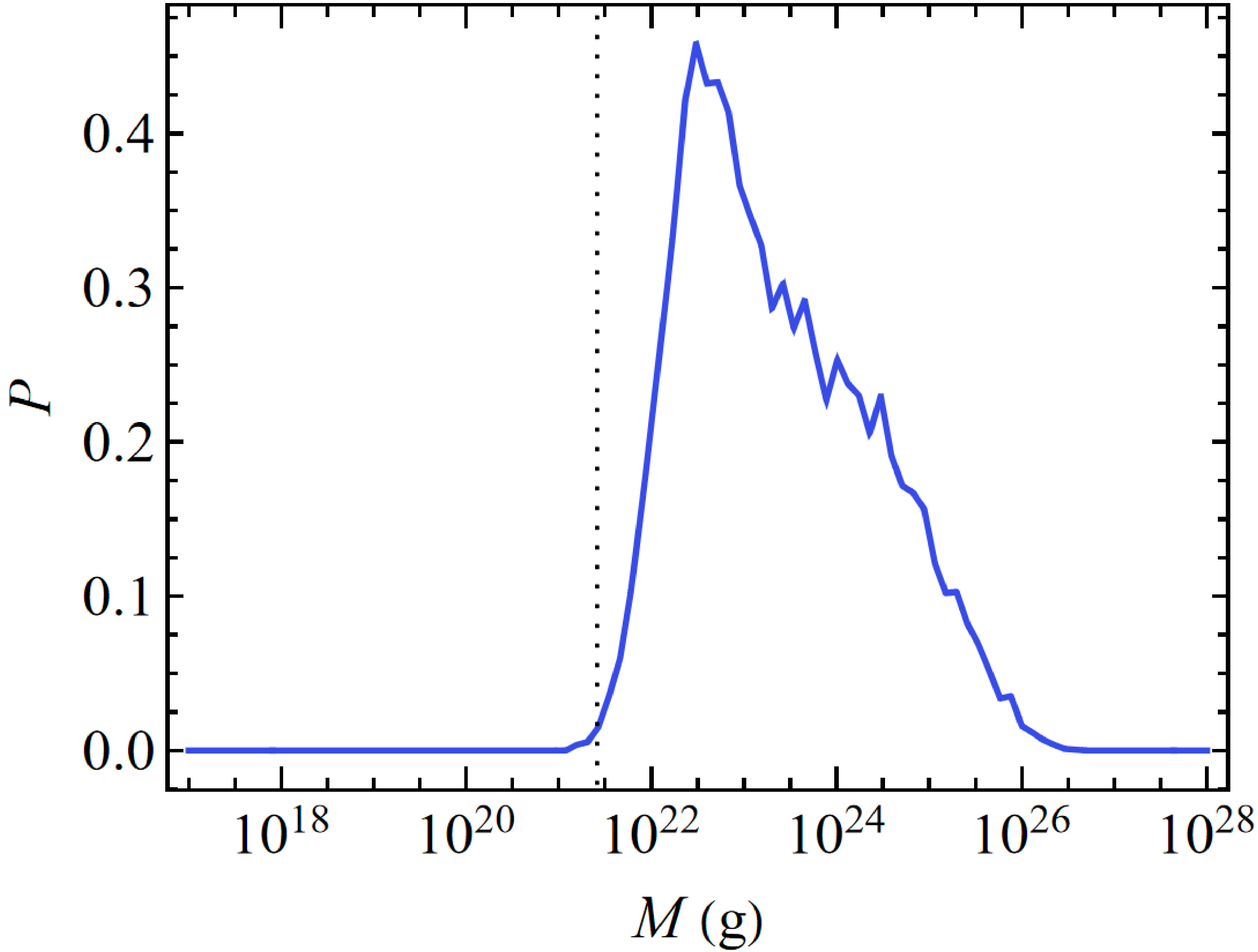
P
0.0
0.1
0.2
0.3
0.4
10^{18}
10^{20}
10^{22}
10^{24}
10^{26}
10^{28}
M (g)

Alternative collapse measure: averaged compaction function

$$R = are^{\zeta}$$

$$\begin{aligned}\bar{\mathcal{C}}(r) &\equiv \frac{3}{R(r)^3}\int_0^{R(r)} \mathrm{d}\tilde{R}\,\tilde{R}^2\mathcal{C} \\ &= -\frac{2}{r^3 e^{3\zeta(r)}}\int_0^r \mathrm{d}\tilde{r}\,\tilde{r}^2 e^{3\zeta}[2\tilde{r}\zeta' + 3(\tilde{r}\zeta')^2 + (\tilde{r}\zeta')^3]\end{aligned}$$